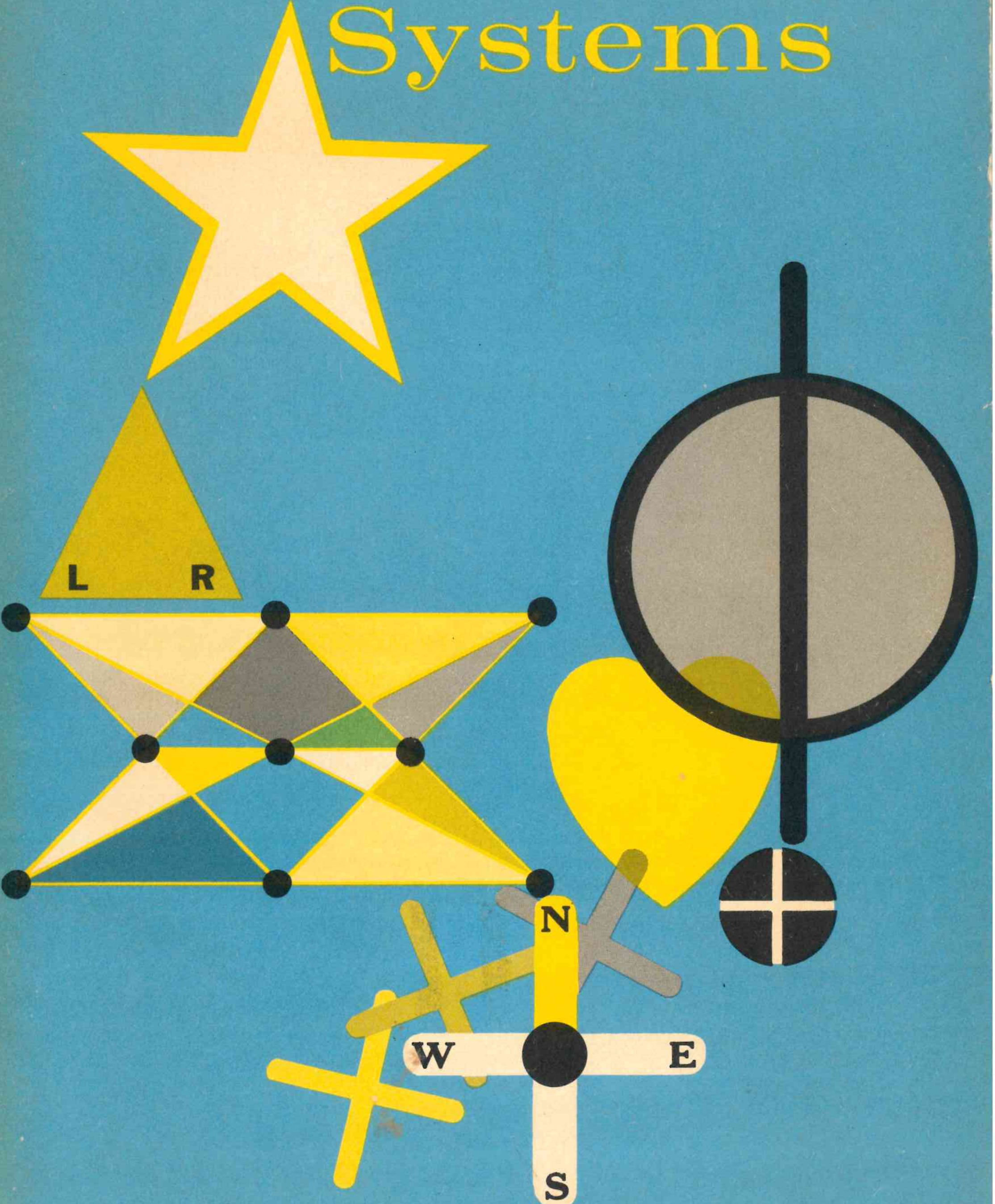
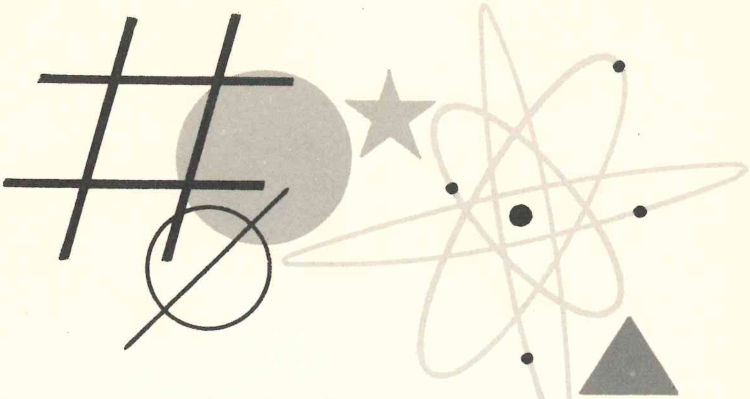
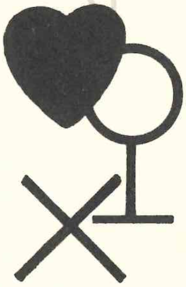
FINITE Mathematical Systems



EXPLORING MATHEMATICS ON YOUR OWN



The magnificent conception of mathematics
as the study of all abstract logical systems
or abstract mathematical sciences and their
concrete interpretations or applications
really justifies the statement that mathematics
is basic to every subject forming part of
the search for truth.
Fundamentals of Mathematics,
Moses Richardson



BEFORE YOU BEGIN TO READ

Reading about mathematics can be as exciting an adventure as reading a mystery story or exploring a cave. There are many surprises, puzzles, tricks, and interesting ideas in mathematics. If you will do some exploring in mathematics by yourself, you will enjoy discovering new ideas. One of the most interesting topics in mathematics is the study of finite mathematical systems.

You will probably need to read this booklet on finite systems differently from the way you would read a story, for example. To begin with, you should read it slowly. Don't be surprised if you don't quite understand every sentence or every paragraph the first time. But have patience. Get into the habit of reading mathematics with a pencil and paper within easy reach. Don't hesitate to use them at every opportunity. If you work the exercises and complete the problems, it will be easier for you to understand what you read.

We hope that this booklet will enable you to share the pleasure others have had in exploring mathematics.

EXPLORING MATHEMATICS ON YOUR OWN

FINITE MATHEMATICAL SYSTEMS

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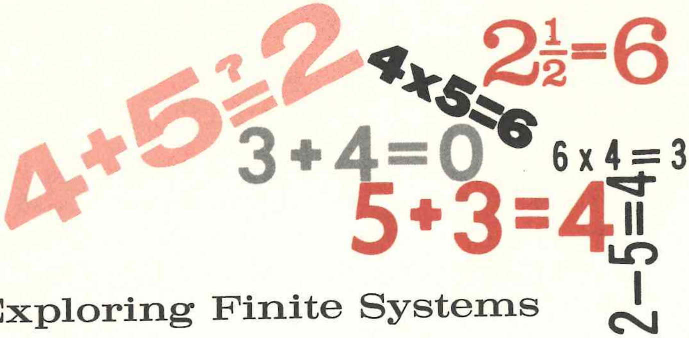


Exploring Finite Systems in Mathematics

Mathematics Is the Exact Science — or Is It?

So you think you know arithmetic? How much is 4 + 5? If you answer 9, you could be wrong! There is an arithmetic system in which $4 + 5 = 2$. Other curious arithmetic facts in this system are $3 + 4 = 0$, $5 \div 3 = 4$, and $4 \times 5 = 6$. This is only one of the systems of arithmetic that you will study in this booklet. If such mathematical systems seem strange to you, you may be surprised to learn that you use systems similar to the one mentioned above several times a day. It can be called clock arithmetic, modular arithmetic, arithmetic of residue classes, or the study of *finite mathematical systems.*

Learning about finite mathematical systems is one of the most interesting studies in mathematics. In general, a finite system is one which has a specific number of elements. The number of elements must be zero or a counting number. This type of system is in contrast to the system of natural numbers which has an unlimited (*infinite*) number of elements.

Examples of sets with a finite number of elements are:

A dozen apples
The numerals on a clock
Your weight in pounds
The odd numbers between 0 and 20
Zero and one

Examples of sets with an infinite number of elements are:

The set of all fractions
The set of all squares
The number of lines in a plane

In a similar way, some mathematical systems are finite and others are infinite.

In this booklet, we will explore some ideas concerned with finite mathematical systems. Through exploring this topic, you will learn to know and to appreciate *structure* in mathematics. You will also develop a better understanding of the logic of mathematics and learn how to use this logic in your study of mathematics.

Monday + Friday = Sunday???

2+6=1

Seven Days Make a Week — and Also an Interesting Finite Arithmetic System

"What day is today?" inquired Brian. "Well, it's just seven days until the big football game which is on a Friday," answered Greg, "so today has to be Friday."

Greg had observed an interesting idea concerning the days of the week. Seven days from any Friday always results in another Friday.

By assigning specific numbers to each of the seven days of the week, an interesting finite arithmetic system can be developed.

S	M	T	W	Th	F	S
1	2	3	4	5	6	7

Weekday Chart

From Friday, represented by 6 as the sixth day of the week, count Saturday, Sunday, Monday, Tuesday, Wednesday, Thursday, and Friday. You can readily see that Friday plus seven days gives Friday once again. Similarly, seven days from Monday gives Monday once again.

What day comes three days after Friday? You are correct if your answer is Monday. If the number 6, representing the day of Friday, is added to 3 by counting on the weekday chart, the numerical result is 2, which represents Monday.

What day is four days past Wednesday? Or by using the number assigned to Wednesday, $4 + 4 =$ what number on the weekday chart? Since four days from Wednesday is always Sunday, from the assigned numbers we can readily see that $4 + 4 = 1$.

We can illustrate another example of this finite system by considering a dial marked with the numerals 0, 1, 2, 3, 4, 5, and 6.

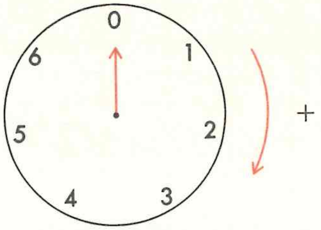


Figure 1

How can we use this dial or clock to get these results?

$$2 + 4 = 6$$
$$5 + 6 = 4$$
$$5 + 2 = 0$$
$$2 + 6 = 1$$
$$3 + 4 = 0$$

How are these answers obtained? You may have noticed that they are obtained in the same manner that we used in the weekday chart discussed earlier. The elements have merely been placed on the "clock" and the element 0 used in place of 7.

We add the elements in this system by rotating clockwise from zero the number of spaces indicated by each addend.

In this system, to obtain the answer to the addition problem $2 + 4$, rotate two spaces clockwise from zero and then four more spaces, ending at 6. Thus, $2 + 4 = 6$. If we were to express this idea in terms of days of the week, we would say that four days from Monday is Friday, or Monday + Wednesday equals Friday.

On the clock shown in Figure 1, the number following six is zero. Thus, when one revolution is completed, we simply begin the count or pattern again.

Zero can also be interpreted to mean "no rotation." Thus, $6 + 0 = 6$, $0 + 4 = 4$, and $3 + 0 = 3$. Since the system has seven elements and the pattern continually repeats, this finite system is said to be a *modular 7 arithmetic*, or simply arithmetic mod 7.

Using the definition for addition in mod 7 arithmetic, complete the addition table in Figure 2.

Addend

Addend

+	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

Figure 2

Are Answers in Mathematics Always Possible?

Whenever two counting numbers are added, such as $3 + 5$, the result is always a unique counting number — in this case, 8. The sum of 3 and 5 is never any other number. The sum of two counting numbers is always another counting number. So we say the counting numbers are *closed* for the addition operation. You can't get outside the set of counting numbers by adding two counting numbers. Not all sets of numbers are closed for addition. Consider the set of odd numbers. Is the set of odd numbers closed for addition? Since the sum of the odd numbers 3 and 5 is 8, an even number, we say the set of odd numbers is not closed under addition.

Next let's examine the operations of multiplication, division, and subtraction. Are the counting numbers closed under these operations? Is the operation of division closed for counting numbers? The quotient $8 \div 2 = 4$ satisfies the law of closure, but $8 \div 3$ is $2\frac{2}{3}$, which is not a counting number.

You have seen that obtaining answers for a given set of elements under a defined operation is not always possible, if we restrict the answers to elements within the set.

How does closure apply to the finite system of clock arithmetic mod 7? The completed table for addition in mod 7 arithmetic is shown below in Figure 3.

Addend

	+	0	1	2	3	4	5	6
Addend	0	0	1	2	3	4	5	6
	1	1	2	3	4	5	6	0
	2	2	3	4	5	6	0	1
	3	3	4	5	6	0	1	2
	4	4	5	6	0	1	2	3
	5	5	6	0	1	2	3	4
	6	6	0	1	2	3	4	5

Figure 3

Examine the addition table carefully. Note that for all possible combinations of elements in this system under the operation of addition, such as $3 + 4 = 0$, a unique answer results which is also an element in the system. Regardless of the elements substituted into the general statement $a + b$, the result is a unique answer c, which is also one of the seven elements. Thus, the operation of addition is closed for our finite system mod 7.

EXERCISE SET 1

Clock Arithmetic — Mod 12

1. On a regular 12-hour clock, what time is it 4 hours after 9:00 A.M.? Six hours after 7:00 P.M.?

On the regular clock, $9 + 4 = 1$ and $7 + 6 = 1$. By defining the operation of addition on the clock as rotating clockwise the number of spaces indicated by each addend, complete the following table in Figure 5 for addition for the finite clock arithmetic system mod 12.

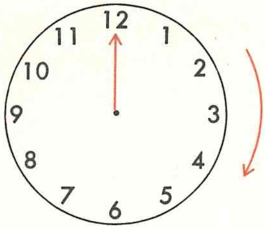


Figure 4

Addend

Addend

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4										
3	4		6									
4	5			8								
5	6				10							
6	7					12						
7	8						2					
8	9							4				
9	10								6			
10	11									8		
11	12										10	
12	1											12

Figure 5

2. How many addition combinations are possible in the mod 12 system? Is addition closed in the mod 12 system of clock arithmetic?

3. In the mod 12 system are the following statements true?
   a. $6 + 7 = 7 + 6$
   b. $8 + 12 = 12 + 8$
   c. $1 + 9 = 9 + 1$
   d. $11 + 5 = 5 + 11$
   e. $12 + 2 = 2 + 12$

Finite Systems and Properties of Mathematics

Certainly 7 + 6 = 6 + 7 — or Does It?

John challenged Randy to a game of "Opposite Order." "How do we play it?" inquired Randy. "I do two things and then you do the same two things in opposite order," answered John.

Randy accepted the challenge. Would you have accepted John's challenge?

Suppose John first unwrapped a candy bar, then ate it. Do you feel you could meet John's challenge in opposite order? What if John picked up a tack from a chair, then sat down? Obviously, the opposite order would make considerable difference.

Consider the problem of adding the counting numbers 6 and 7. Does the changing of the order for adding the addends make any difference in the result? If your answer was no, you are correct, since both $7 + 6$ and $6 + 7$ are equal to 13. What about the changing of order in subtraction, such as in the problem $7 - 6 = 6 - 7$? It should be apparent to you that with subtraction the order of the subtraction cannot be changed without changing the answer except in special cases, such as $5 - 5$ or $0 - 0$.

When for all possible cases, such as $6 + 3 = 3 + 6$, the change of the order of adding the addends does not change the sum, we say the operation is *commutative*. In the cases of subtraction and division of counting numbers, $5 - 3 \neq 3 - 5$ and $8 \div 4 \neq 4 \div 8$, obviously the commutative property does not apply.

Thus, you can see that some systems are commutative for a particular operation, but are not necessarily commutative for all operations.

Let's return once again to the mod 7 system of clock arithmetic. Is this system commutative? Does $6 + 2 = 2 + 6$ in mod 7? Check the addition table completed earlier to answer this question.

By examination of the addition table for mod 7, you note that $6 + 2 = 1$ and $2 + 6 = 1$.

In the following examples, the symbol $\overset{?}{=}$ is used to form a question. For instance, $a + b \overset{?}{=} b + a$ should be read, "Is a plus b equal to b plus a?"

a. $5 + 4 \overset{?}{=} 4 + 5$
b. $2 + 1 \overset{?}{=} 1 + 2$
c. $0 + 6 \overset{?}{=} 6 + 0$
d. $1 + 3 \overset{?}{=} 3 + 1$
e. $2 + 5 \overset{?}{=} 5 + 2$
f. $3 + 0 \overset{?}{=} 0 + 3$
g. $6 + 5 \overset{?}{=} 5 + 6$
h. $4 + 1 \overset{?}{=} 1 + 4$

What might you conclude about the commutative property from the examples above in regard to the mod 7 system?

How many examples would you have to work in order to be absolutely certain that a system was commutative? If you examined each possible combination except one, the one not attempted might be contrary to all previous findings. Therefore, every possibility would have to be explored.

There is an interesting short cut for checking a system for commutativity which we will point out. Examine carefully the completed table for addition in mod 7 clock arithmetic, Figure 6.

		Addend						
	+	0	1	2	3	4	5	6
Addend	0	0	1	2	3	4	5	6
	1	1	2	3	4	5	6	0
	2	2	3	4	5	6	0	1
	3	3	4	5	6	0	1	2
	4	4	5	6	0	1	2	3
	5	5	6	0	1	2	3	4
	6	6	0	1	2	3	4	5

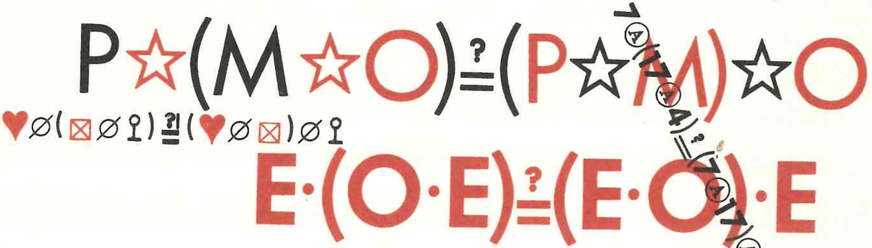
Figure 6

Do you note any patterns in the table? Draw a diagonal line from the top left-hand corner to the bottom right-hand corner of the addition table as shown in Figure 6.

If we fold the table with respect to the diagonal, all numbers will correspond. That is, the answer 6 in the top right-hand corner of the table will cover the answer 6 in the lower left-hand corner. It is said that the table is *symmetric* with respect to the diagonal.

The answer 6 in the upper right-hand corner is the result of adding $0 + 6$. The 6 in the lower left-hand corner is the result of $6 + 0$. In similar manner, you should note that the corresponding numbers of the folded table are the results of an exchange of addends such as $5 + 1 = 1 + 5$, $4 + 3 = 3 + 4$, or $2 + 1 = 1 + 2$.

A timesaver for determining whether or not an operation is commutative is simply to check the completed table for that operation. If it is symmetric with respect to the diagonal as discussed earlier, the operation is commutative. We can be sure that for all elements a and b, $a + b = b + a$ in the mod 7 system.



The Associative Principle — an Important Idea

Add the counting numbers 1, 2, and 3, taking two numbers at one time as shown below.

$$1 + (2 + 3) = ?$$
$$(1 + 2) + 3 = ?$$

In each combination above, the sum is 6. In the case of adding 5, 6, and 7, the same sum results when we associate the 6 with 7 and then add this result to 5 as when we associate the 6 with the 5 and add the result to 7.

$$5 + (6 + 7) \overset{?}{=} (5 + 6) + 7$$
$$5 + (13) \overset{?}{=} (11) + 7$$
$$18 = 18$$

For another example we might try $13 + (7 + 3) \overset{?}{=} (13 + 7) + 3$. Since both sides are equal to 23, we say that the operation is *associative* for this example.

Not all operations are associative as is evident from these two examples using division and subtraction of counting numbers.

Example 1: $8 \div (4 \div 2) \overset{?}{=} (8 \div 4) \div 2$

$$8 \div (2) \overset{?}{=} (2) \div 2$$

$$4 \neq 1$$

Example 2: $6 - (3 - 2) \overset{?}{=} (6 - 3) - 2$

$$6 - (1) \overset{?}{=} (3) - 2$$

$$5 \neq 1$$

If the operation symbol Ⓐ means to find the average of two counting numbers, we find another operation where the associative property does not apply.

If $6 \textcircled{A} 2$ means the average of 6 and 2, the answer is 4. Does the commutative property apply to the example which follows?

$$6 \textcircled{A} (2 \textcircled{A} 2) \overset{?}{=} (6 \textcircled{A} 2) \textcircled{A} 2$$

$$6 \textcircled{A} (2) \overset{?}{=} (4) \textcircled{A} 2$$

$$4 \neq 3$$

Check the following examples for associativity using the operation as defined above.

a. $5 \textcircled{A} (3 \textcircled{A} 3) \overset{?}{=} (5 \textcircled{A} 3) \textcircled{A} 3$

b. $16 \textcircled{A} (9 \textcircled{A} 5) \overset{?}{=} (16 \textcircled{A} 9) \textcircled{A} 5$

c. $1 \textcircled{A} (2 \textcircled{A} 6) \overset{?}{=} (1 \textcircled{A} 2) \textcircled{A} 6$

d. $8 \textcircled{A} (8 \textcircled{A} 16) \overset{?}{=} (8 \textcircled{A} 8) \textcircled{A} 16$

Check the addition table once again for the mod 7 system of clock arithmetic.

What is the sum of $6 + 3 + 4$? The correct answer is 6 since $3 + 4 = 0$ and $0 + 6 = 6$ in this finite system. Would the sum be different if 6 and 3 were added first? Since $6 + 3$ is 2 and $2 + 4$ is 6, the sums are equal, and we say the associative property holds true for this example. Thus, $6 + (3 + 4) = (6 + 3) + 4 = 6$. In considering $6 + (3 + 4)$ from the standpoint of days of the week, we are saying that four days from Tuesday (3) is Saturday (0) and 0 days from Friday (6) is Friday (6) again.

In a similar manner, (6 + 3) + 4 in a weekday chart means 3 days from Friday (6) which is Monday (2) and 4 days from Monday is Friday (6).

Check the following in the mod 7 system for associativity.

a. $4 + (3 + 2) \overset{?}{=} (4 + 3) + 2$
b. $5 + (6 + 0) \overset{?}{=} (5 + 6) + 0$
c. $1 + (4 + 5) \overset{?}{=} (1 + 4) + 5$
d. $4 + (5 + 5) \overset{?}{=} (4 + 5) + 5$

Other Important Properties — Identities, Inverses, and Distributivity

We have discussed earlier the fact that 7 added to the number which represents any day of the week, such as Sunday, returns you to that same day.

In our system of clock arithmetic mod 7, seven has been replaced by 0.

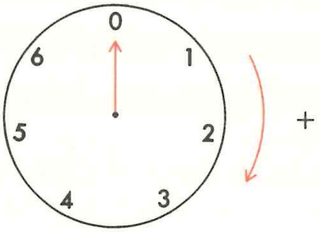


Figure 7

In all cases as 6 + 0, 5 + 0, 4 + 0, 3 + 0, 2 + 0, 1 + 0, and 0 + 0, the sums are identical to the element which was added to 0. We say that 0 is the *identity element* for the addition operation, since for any element n, $n + 0 = n$. This is an interesting and important property to consider in the structure of a mathematical system. We will need to consider this property in more detail as we explore other finite systems.

Can you name the identity element for addition in the system of ordinary arithmetic? $13 + \underline{?} = 13$; $\underline{?} + 23 = 23$?

Jim's birthday was only one week away. On Saturday he made a note that in 7 days he would be 16. On Sunday he recorded that his birthday would be in 6 days. On Monday 5 days remained until Saturday when he would become 16. Using the weekday chart below, the following relationships are true.

$7 + 7 = 7$	7 (Saturday) + 7	= Saturday (7)
$1 + 6 = 7$	1 (Sunday) + 6	= Saturday (7)
$2 + 5 = 7$	2 (Monday) + 5	= Saturday (7)
$3 + 4 = 7$	3 (Tuesday) + 4	= Saturday (7)
$4 + 3 = 7$	4 (Wednesday) + 3	= Saturday (7)
$5 + 2 = 7$	5 (Thursday) + 2	= Saturday (7)
$6 + 1 = 7$	6 (Friday) + 1	= Saturday (7)

By substituting 0 for the number 7, the results of the additions are: $0 + 0 = 0$, $1 + 6 = 0$, $2 + 5 = 0$, $3 + 4 = 0$, $4 + 3 = 0$, $5 + 2 = 0$, and $6 + 1 = 0$. Compare these results with the same combinations in addition for mod 7 in the table in Figure 3. When two elements that are combined under an operation result in the identity element for that operation, we say that they are *inverses* of each other.

Since we have discovered that $6 + 1$, $2 + 5$, $3 + 4$, and $0 + 0$ all have the additive identity element of 0 for a sum, we can say that 6 is the additive inverse of 1 and similarly 1 is the additive inverse of 6. Also, 2 and 5 are additive inverses, as are 3 and 4. We can say that 0 is its own additive inverse.

Consider this example for counting numbers $5(6 + 7) = ?$

You can solve this problem in two ways. One way is to add the 6 and 7 and multiply this sum of 13 by 5. The result is 65. Another method you might have discovered is to multiply the 6 by 5 and add this product of 30 to the product resulting from 5×7, which is 35. Thus, $30 + 35 = 65$. By either method, the answers are equal. When $a(b + c) = ab + ac$ for all possibilities in a system, we say that the system is *distributive* for multiplication with respect to addition. Let us consider the following example.

$$6(4 + 3) \overset{?}{=} 6 \cdot 4 + 6 \cdot 3$$
$$6(7) \overset{?}{=} 24 + 18$$
$$42 = 42$$

Since both products equal 42, we can say that the *distributive principle* holds true for this example.

EXERCISE SET 2

Applying Basic Principles to Finite Systems

1. Do the following examples satisfy the associative principle in mod 12 arithmetic?

a. $9 + (8 + 7) = (9 + 8) + 7$
b. $10 + (10 + 12) = (10 + 10) + 12$
c. $10 + (3 + 1) = (10 + 3) + 1$
d. $2 + (2 + 4) = (2 + 2) + 4$

2. What is the identity element for addition in the system, mod 12? Give several examples which exemplify the identity element.

3. Give the additive inverse of each of the given elements in the mod 12 system:

a. The inverse of 1 is ____.
b. The inverse of 2 is ____.
c. The inverse of 3 is ____.
d. The inverse of 5 is ____.
e. The inverse of 10 is ____.

Does each element in the system have an additive inverse?

4. Consider a system of clock arithmetic with two elements, or numbers 0 and 1, which are placed on a dial or clock as shown below.

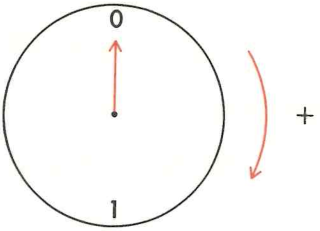


Figure 8

Complete the addition table for this mod 2 system.

+	0	1
0		
1		

Figure 9

It is noted that after two steps are taken, the pattern is repeated. What is the additive identity element for the system?

5. Answer the following questions in regard to the mod 2 system in Problem 4.

a. Is the system closed for the operation of addition?

b. Since $0 + 1 = 1 + 0$ in this system, what principle holds true?

c. Solve each of the following exercises to check for the associative principle.

(1) $1 + (1 + 1) \overset{?}{=} (1 + 1) + 1$

(2) $0 + (0 + 1) \overset{?}{=} (0 + 0) + 1$

(3) $1 + (1 + 0) \overset{?}{=} (1 + 1) + 0$

(4) $1 + (0 + 0) \overset{?}{=} (1 + 0) + 0$

(5) $0 + (1 + 0) \overset{?}{=} (0 + 1) + 0$

d. Note that $1 + 0 = 1$, $0 + 0 = 0$, and $0 + 1 = 1$. What is the identity element for addition?

e. Does each element in the system have an additive inverse? What numbers must be added to the given elements to obtain the identity element of 0 as a sum?

(1) $? + 0 = 0$

(2) $? + 1 = 0$

(3) $0 + ? = 0$

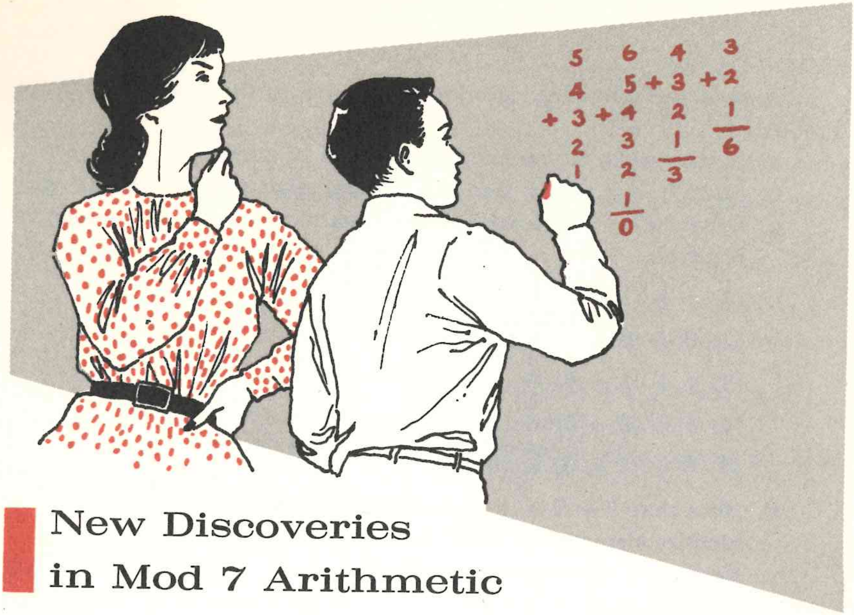
(4) $1 + ? = 0$

6. Defining the operation Ⓢ as moving from one season to the next one in order of the seasons as they occur, complete an addition table for Summer, Fall, Winter, and Spring. For example, three seasons from Fall (2) is Summer (1); thus, $2 + 3 = 1$.

Su	F	W	Sp
1	2	3	4

Ⓢ	1	2	3	4
1	2			
2		4	1	
3			2	
4				4

Figure 10



New Discoveries in Mod 7 Arithmetic

Multiplication in Finite Arithmetics

When does 5 × 6 = 2? You may think that this is never possible. But it is if we are computing with numbers in mod 7 clock arithmetic.

If we define the example 5 × 6 to mean five sixes or 6 + 6 + 6 + 6 + 6, we can find the answer by checking the addition table for *addition* in mod 7.

Example: 5 × 6 = ?

5 × 6 = (6 + 6) + (6 + 6) + 6	by definition of multiplication
= (5) + (5) + 6	adding (6 + 6) mod 7
= (3) + 6	adding (5) + (5) mod 7
= 2	adding (3) + 6 mod 7

We have to determine what 0 · 3 and 3 · 0 are to mean in order to be able to solve these problems. Since we have defined 3 · 4 to mean 4 + 4 + 4, 3 · 0 means 0 + 0 + 0 which equals 0 and 0 · 3 means "no threes" or 0.

You might wonder if a table for multiplication can be developed as was done with mod 7 addition. Such a table is shown in Figure 11.

Factor

Factor

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1					6
2	0		4				5
3	0			2			
4	0				2	6	
5	0	5		1		4	
6	0						1

Figure 11

Complete the table and then determine if the operation of multiplication is closed for mod 7 clock arithmetic. It is noted that 1 × 4 = 4, 1 × 6 = 6, 5 × 1 = 5, and 0 × 1 = 0. In fact for any element n, $n \times 1 = n$. We say that 1 is the identity element for multiplication, or the *multiplicative identity*.

If the product of two elements results in 1, we say they are *multiplicative inverses*. Sometimes we use the term *reciprocals*. It means exactly the same. Examples of inverses in mod 7 multiplication are 5 and 3, since 5 × 3 = 1.

Applications of Modular Multiplication

You no doubt have gone through turnstiles at bus depots, cafeterias, and train stations. Consider a turnstile with four arms; one arm is colored red. Each time one person passes through the turnstile, the red arm moves one quarter of a turn.

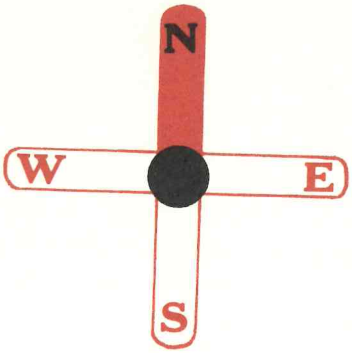


Figure 12

We could assume the following numbers to correspond to the positions of the arms of the turnstile.

North	East	South	West
0	1	2	3

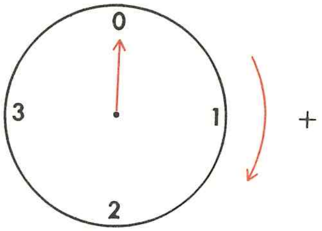


Figure 13

If the red arm was pointing east and three persons passed through the turnstile, what would be the position of the red arm? Since the east position corresponds to 1 in mod 4 arithmetic, $1 + 3 = 0$, and the red arm would now be in the north position or position 0.

If the red arm was in the east position and two persons passed through, the problem to be solved is $1 + 2$, which equals 3. Therefore, we see that the red arm would be in the west position.

What would be the result if the red arm was in the west position and three persons passed through? Since west is at the 3 position, $3 + 3 = 2$, and the red arm would be in the south position. Consider the problem when the red arm is in the west position and 9 persons pass through. The number nine is not an element of our mod 4 system. We could arrive at the answer by beginning at the west position (3) and by counting clockwise 9 spaces.

However, since we are working with a mod 4 arithmetic, once we reach four, the pattern or count begins over; we "cast out" the four and begin again. In the case of 9, we count 9 as follows: 1, 2, 3, 4 (cast out 4), proceed to 1, 2, 3, 4 (cast out 4), and then proceed to 1. On the mod 4 clock, counting 9 spaces from 0 is the same as counting 1 space. We can see, then, that if the red arm of the turnstile was in the west position (3) and 9 persons passed through, the problem can be solved either by beginning at 3 and counting 9 spaces or by casting out fours in the number 9 and adding 1 to 3. Our answer is that the red arm would be at 0, or the north position, following the passage through of the 9 persons.

Suppose three men went through the turnstile, with the red arm originally pointing east before the first man passed through. Each man in turn went out an exit and then returned to go through the turnstile one more time. What would be the position of the red arm after this occurred? The result of the three men each passing through twice would leave the red arm in the west position. $3 \cdot 2$ in mod 4 arithmetic is equal to 2, our position for south. Since the red arm was at 1 (east) to begin, $2 + 1 = 3$, which shows the red arm in the 3, or the west, position.

Now let's consider a more practical application of multiplication in the finite system, mod 7.

John's regular lunch allowance lasts for 4 days. On one Friday night, rather than the regular allowance, John's father gave him five times the regular amount. On what day of the week will John need lunch money again?

Of course, you could find the answer by counting, but multiplication in mod 7 provides a means for arriving at a quick answer. John's usual allowance lasts 4 days and he received 5 times the regular amount.

In mod 7, $4 \times 5 = 6$. (See Figure 11.)

Since we begin with Friday, the night the allowance was received, we could let this day be 0, Saturday 1, Sunday 2, and so forth as shown below.

F	S	S	M	T	W	Th
0	1	2	3	4	5	6

Weekday Chart

From this weekday chart, it is seen that 6 corresponds to Thursday. We now can see that John will have to have lunch money next on a Thursday night if he is to have lunch money for Friday the following day.

Actually, finite systems of modular arithmetic are finding many practical uses in such fields as nuclear energy and other fields of research. For example, the rings of electrons in atoms can be related to modular number systems.

EXERCISE SET 3

Further Work with Modular Systems

1. Check the completed table for multiplication in mod 7 (Figure 11). Is there symmetry in the table? What principle does this suggest concerning the operation of multiplication in mod 7?

2. Follow the steps in the example which follows. This is a check for associativity in mod 7 multiplication. Check the other exercises for the associative principle on your own.

Example:

$$3 \cdot (4 \cdot 5) \overset{?}{=} (3 \cdot 4) \cdot 5$$
$$3 \cdot (5+5+5+5) \overset{?}{=} (4+4+4) \cdot 5$$
$$3 \cdot (6) \overset{?}{=} (5) \cdot 5$$
$$6 + 6 + 6 \overset{?}{=} 5 + 5 + 5 + 5 + 5$$
$$4 = 4$$

Since both sides equal 4, the associative principle holds true for the example.

a. $2 \cdot (5 \cdot 3) \overset{?}{=} (2 \cdot 5) \cdot 3$

b. $6 \cdot (4 \cdot 1) \overset{?}{=} (6 \cdot 4) \cdot 1$

c. $3 \cdot (3 \cdot 4) \overset{?}{=} (3 \cdot 3) \cdot 4$

d. $5 \cdot (3 \cdot 2) \overset{?}{=}$

e. $6 \cdot (6 \cdot 5) \overset{?}{=}$

3. What is the identity element for the mod 7 system for multiplication?

4. From the completed multiplication table for mod 7, check to see what combinations result in the identity element 1.

The multiplicative inverse of

6 is ____.
5 is ____.
4 is ____.
3 is ____.
2 is ____.
1 is ____.

5. The distributive principle is stated in general terms as follows:

$$a(b + c) = ab + ac$$

Use your knowledge of addition and multiplication in mod 7 arithmetic to determine if the distributive principle holds true for the following exercises.

a. $3(2 + 4) \stackrel{?}{=} 3 \cdot 2 + 3 \cdot 4$

$$3(6) \stackrel{?}{=} (2 + 2 + 2) + (4 + 4 + 4)$$

$$6 + 6 + 6 \stackrel{?}{=} (6) + (5)$$

$$4 = ?$$

b. $2(5 + 3) \stackrel{?}{=} 2 \cdot 5 + 2 \cdot 3$

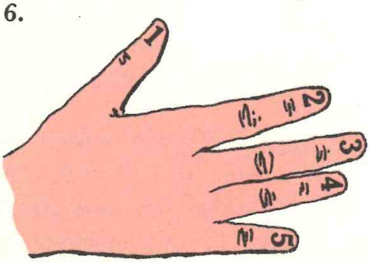
c. $0(1 + 5) \stackrel{?}{=} 0 \cdot 1 + 0 \cdot 5$

d. $3(4 + 6) \stackrel{?}{=} 3 \cdot 4 + 3 \cdot 6$

e. $4(3 + 3) \stackrel{?}{=}$ Complete

f. $5(5 + 1) \stackrel{?}{=}$ Complete

6.



Start with your thumb as number 1 and assign a number 2, 3, 4, and 5 to each finger in consecutive order. By counting, $1 + 4 = 5$. When you reach the little finger, or 5, continue counting again with the thumb. Thus, $3 + 4 = 2$. Using this idea, complete the addition table in Figure 14. Then, using the definition of multiplication of $3 \cdot 4$ to mean three fours or $4 + 4 + 4$, complete the multiplication table in Figure 15 for this system.

Addend

Addend

+	1	2	3	4	5
1	2				
2		4			
3			1		
4				3	
5					5

Figure 14

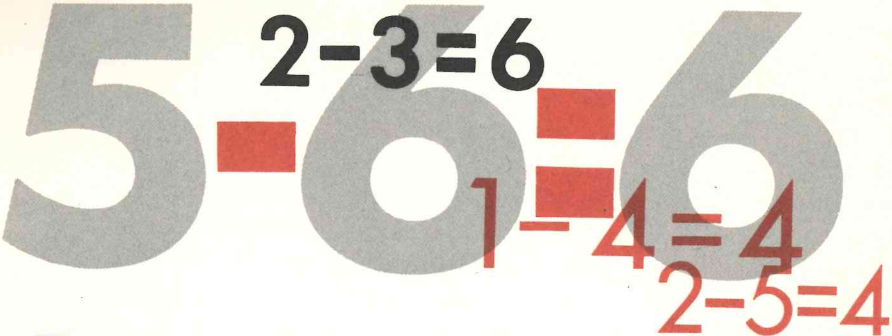
factor

factor

×	1	2	3	4	5
1	1	2	3	4	5
2	2				
3	3				
4	4				
5	5				

Figure 15

7. What is the modular number of the arithmetic system in Exercise 6? Does it have an identity element for addition? For multiplication?



Exploring Other Operations in Finite Systems

Eliminating the Negative

Which of the following problems have answers which are correct? $4 - 1 = 3$, $1 - 4 = 4$, $2 - 3 = 6$, $6 - 5 = 1$, and $2 - 5 = 4$. If your answer to this question is the first and fourth problems, you are correct, but only partly so. All answers are correct for subtraction in the mod 7 system of clock arithmetic.

The operation of subtraction $(-)$ in the mod 7 system of clock arithmetic, just as in ordinary arithmetic, is defined in terms of addition.

In ordinary arithmetic, using counting numbers, the answer to the problem $13 - 4$ is obtained by finding the number which added to 4 gives 13. Since $4 + 9 = 13$, the number 9 is the answer to the problem $13 - 4 = 9$.

You have used this process many times in checking answers in subtraction. In checking, we add the difference to the subtrahend to see if this sum equals the minuend. We say that subtraction is the *inverse operation* of addition.

Using the elements in mod 7, solve the subtraction problem $6 - 5 = ?$ What element added to 5 gives 6? From the addition table for mod 7, $5 + 1 = 6$. Therefore, we can use a 1 in place of the question mark, and $6 - 5 = 1$.

In the system of ordinary arithmetic, we cannot subtract $5 - 6$ without obtaining a negative answer. In mod 7 subtraction, negative numbers are not needed. In the problem $5 - 6 = ?$, we are looking for an element which when added to 6 gives 5. The addition table reveals that the sum of $6 + 6$ is 5. Thus, 6 can replace the ? and $5 - 6 = 6$.

You'll find other subtraction problems in this system quite different and interesting. Consider the problem $4 - 5 = n$, where n represents the answer. By our definition of subtraction, $n + 5$ must equal 4. What number added to 5 in mod 7 results in 4? From the addition table, we readily see that $n = 6$, and thus, $4 - 5 = 6$.

Does $1 - 4 = 4 - 1$ in this finite system? In considering the problem $1 - 4$, what element added to 4 results in 1? Since $4 + 4 = 1$, $1 - 4 = 4$. In similar manner, the problem $4 - 1$ means you are to find a number which when added to 1 equals 4. Since $1 + 3 = 4$, $4 - 1$ must be 3. You can see, therefore, that since $1 - 4 \neq 4 - 1$, the commutative principle does not hold true for the subtraction operation.

Assigning numbers 1 through 7 to the days of the week as shown below, we might ask what day was four days before last Saturday, or $7 - 4 = ?$ By counting 4 days backward from Saturday, it is seen that the answer is Tuesday (3). Therefore, $7 - 4 = 3$. In a similar manner, $6 - 7 = 6$, and $5 - 6 = 6$.

S	M	T	W	Th	F	S
1	2	3	4	5	6	7

Weekday Chart

If we placed these elements on the clock and used 0 in place of 7, as we have done previously, we can relate the subtraction operation directly to the mod 7 system of clock arithmetic that we have explored.

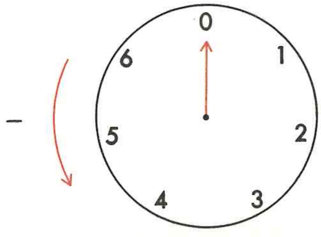


Figure 16

How can the subtraction problem 3 − 5 be solved on the mod 7 clock? You could count 5 spaces *backward* from 3 on the clock and arrive at the answer 5. Or, by use of the definition of subtraction, what number added to 5 gives 3? By counting or by use of the addition table, you see that 5 + 5 = 3. Therefore, it is apparent that 3 − 5 = 5.

EXERCISE SET 4

Extending Concepts in Subtraction

1. Work the following exercises in subtraction for the mod 7 system and then complete the subtraction table in Figure 16.

a. 3 − 2 =
b. 2 − 3 =
c. 6 − 0 =
d. 0 − 6 =
e. 5 − 2 =
f. 2 − 5 =
g. 6 − 6 =
h. 4 − 0 =

Complete the table for subtraction, mod 7.

Subtrahend

Minuend

−	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

Figure 17

2. Check the completed table in Figure 17. Is the operation of subtraction closed in mod 7?

3. The example below checks subtraction in this system for the associative principle. Follow the steps; then check the two following exercises for associativity.

Example: Does $5 - (6 - 3) = (5 - 6) - 3$?

$$5 - (3) \overset{?}{=} (6) - 3$$
$$2 \neq 3$$

Since the right side equals 3 and the left side equals 2, the associative principle does not hold true.

a. $6 - (3 - 1) \overset{?}{=} (6 - 3) - 1$
b. $4 - (6 - 5) \overset{?}{=} (4 - 6) - 5$

4. Check the following exercises for the distributive principle for multiplication with respect to subtraction.

a. $6(5 - 3) \overset{?}{=} 6 \cdot 5 - 6 \cdot 3$
$\quad 6\ (2) \overset{?}{=} 2 - 4$
$\quad 5 = ?$
b. $3(2 - 1) \overset{?}{=} 3 \cdot 2 - 3 \cdot 1$
c. $5(4 - 3) \overset{?}{=} 5 \cdot 4 - 5 \cdot 3$
d. $6(0 - 4) \overset{?}{=} 6 \cdot 0 - 6 \cdot 4$
e. What do the answers for exercises a, b, c, and d suggest in regard to the distributive principle?

5. Using the definition of subtraction that $a - b$ means an answer c such that $c + b = a$, complete the subtraction table for the mod 6 system of clock arithmetic.

Subtrahend

	−	0	1	2	3	4	5
Minuend	0						
	1						
	2						
	3						
	4						
	5						

Figure 18

Are Fractions Necessary?

Is $\frac{1}{4} = 2$ or $\frac{1}{3} = 5$? Believe it or not, in the finite system of mod 7 clock arithmetic, these answers are possible. In fact, if you continually worked with systems of modular arithmetics, you would have no need for fractions. How is this possible? Let's consider the operation of division. In division of the counting numbers, we check answers by multiplying the quotient by the divisor to see if we get the dividend. Thus, $125 \div 25 = 5$ since $5 \times 25 = 125$.

We also can define the operation of division in clock arithmetic as the *inverse operation* of multiplication. For example, the problem $5 \div 4$ is solved by finding a number, n, such that $n \times 4 = 5$. We already have completed the multiplication table for mod 7, and can use it to find the answer. In mod 7, $3 \times 4 = 5$ and thus $5 \div 4 = 3$. Try another example. In mod 7, what is the answer for $6 \div 3$? Since $2 \times 3 = 6$, the answer for the problem $6 \div 3$ is 2. Each element in the system can be divided evenly by another. No answers for division problems ever have remainders.

If we consider the expression $3 \div 2$ to be the same as $\frac{3}{2}$ and $1 \div 3$ the same as $\frac{1}{3}$, an interesting advantage of modular arithmetics becomes apparent.

$$\frac{3}{2} = 3 \div 2 = 5 \qquad \text{Also, } \frac{1}{3} = 1 \div 3 = 5$$

Fractions are quickly changed to some element of the mod 7 system. An expression such as $\frac{2\frac{1}{4}}{3\frac{1}{3}}$ is solved in a similar manner.

The fraction $\frac{1}{4}$ is equivalent to 2, since in mod 7, $4 \times 2 = 1$. Thus, $2\frac{1}{4}$ is equivalent to $2 + 2$, or 4. The denominator, $3\frac{1}{3}$, is equal to 1 since $\frac{1}{3} = 5$ and $3 + 5 = 1$ in mod 7. Finally, $\frac{4}{1} = 4$, since $1 \times 4 = 4$.

Does each element in the mod 7 system have a reciprocal which is also an element in the system? The reciprocal is the multiplicative inverse. The answer is yes, with one exception. The reciprocal of $4 = \frac{1}{4} = 2$, of $3 = \frac{1}{3} = 5$, of $2 = \frac{1}{2} = 4$, and so on.

Consider the problem $2 \div 6$ in mod 7 arithmetic. In this case, we are to find the answer n such that $n \times 6 = 2$. Since $5 \times 6 = 2$ in mod 7 clock arithmetic, $2 \div 6 = 5$. However, we get a very different result if we change the problem to $6 \div 2$. In the case of $6 \div 2$, the answer is 3, since $3 \times 2 = 6$. It is apparent that $6 \div 2$ is not equal to $2 \div 6$, thus, the commutative principle for division in mod 7 does not hold true.

Proceed with Caution
Especially When Working with Zero

If the reciprocal of 4 is $\frac{1}{4}$, what is the reciprocal of 0? Recall that a reciprocal is that number which gives a product of 1 when multiplied by the original number. In the case of 4, $4 \times \frac{1}{4} = 1$, and we say $\frac{1}{4}$ is the reciprocal of 4. What about the reciprocal of 0? What number multiplied by 0 gives 1? $0 \times 0 = 0$, $0 \times 1 = 0$, $0 \times n = 0$. Since 0 multiplied by any number gives 0, 0 has no reciprocal.

Let's explore this idea further in relation to mod 7 clock arithmetic. What does $0 \div 3$ equal in mod 7 division? Is your answer 0? 3? The correct answer is 0, since $3 \times 0 = 0$ in mod 7. Now consider the problem $3 \div 0$. Do you think 0 is the answer? 3? In the problem $3 \div 0$, we must find a number n such that $0 \times n = 3$. Is this possible in the mod 7 system? By checking the multiplication table for mod 7, it is seen that 0 times any number is always 0. Therefore, an answer for $0 \times n = 3$ does not exist. Thus, the problem $3 \div 0$ cannot be solved.

Does the statement that any number divided by itself equals 1 hold true for 0? Does $0 \div 0 = 1$? 0? The answers are no. Since we need an element n, such that $n \times 0 = 0$ to solve these problems, *any* element in the mod 7 system is a solution of $n \times 0 = 0$, which leads to an *inconsistency* in the system.

Therefore, we say that division by 0 in the system is excluded or undefined. Is division by 0 excluded in ordinary arithmetic?

EXERCISE SET 5

Division Considerations — Mod 7 and Mod 5

1. Complete the division table in Figure 19 for mod 7 clock arithmetic.

Dividend

	÷	0	1	2	3	4	5	6
Divisor	0	—	—	—	—	—	—	—
	1	0	1	2	3	4	5	6
	2	0	4	1	5	2	6	3
	3	0			1			
	4	0				1		
	5	0					1	
	6	0		5			2	1

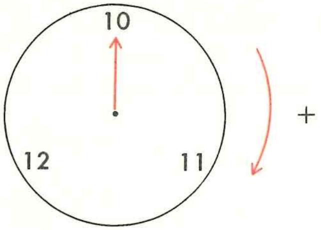
Figure 19

2. Answer the following questions as they apply to division in mod 7.
   a. Is the system closed?
   b. Is the operation commutative? Check several sample problems such as: $6 \div 4 \overset{?}{=} 4 \div 6$
   c. Is the operation associative?
      (1) $2 \div (3 \div 4) \overset{?}{=} (2 \div 3) \div 4$
      (2) $6 \div (3 \div 1) \overset{?}{=} (6 \div 3) \div 1$

3. Is the operation distributive for multiplication?
   a. $3(4 \div 5) \overset{?}{=} 3 \cdot 4 \div 3 \cdot 5$
   b. $6(6 \div 1) \overset{?}{=} 6 \cdot 6 \div 6 \cdot 1$
   c. $3(0 \div 4) \overset{?}{=} 3 \cdot 0 \div 3 \cdot 4$

A Strange New Clock Arithmetic System

When does $10 + 10 = 11$? $12 + 12 = 12$? $11 + 11 = 10$? Not possible? Let's consider the system of clock arithmetic with the 3 elements 10, 11, and 12.



Addend

Addend

+	10	11	12
10	11	12	10
11	12	10	11
12			12

Figure 20

Complete the addition table in Figure 20. $10 + 10$ means begin at 10 and rotate 10 more spaces.

By starting at 10 and counting 10 more spaces clockwise, the result is 11. Thus, in this finite system $10 + 10 = 11$.

Is the completed addition table in Figure 20 symmetric? Does $10 + 11 = 11 + 10$? Since for all elements a and b, $a + b = b + a$, the system is commutative.

We will not explore addition for all properties at this time, but the identity element for this system is quite interesting. Note the following relationships.

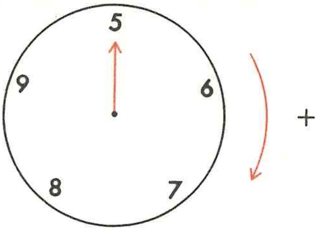
$$10 + (12) = 10$$
$$11 + (12) = 11$$
$$12 + (12) = 12$$

When any element n in the system is added to 12, the identical answer n results. Thus, we see that 12 is the additive identity.

EXERCISE SET 6

Extending Concepts in Clock Arithmetic

1. Complete the addition table for the system of clock arithmetic with elements 5, 6, 7, 8, and 9.



Addend

	+	5	6	7	8	9
Addend	5	5	6	7	8	9
	6					
	7					
	8					
	9	9	5	6	7	8

Figure 21

2. Is the system in exercise 1 closed for addition?

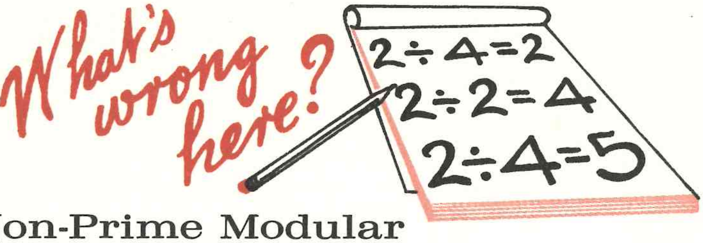
3. Is the system in exercise 1 commutative? Associative?

4. The identity element for addition in exercise 1 is ______.

5. Does each element in the system have an additive inverse? List the additive inverse of each element.

6. Define the multiplication of 5 × 6 in the system with elements 5, 6, 7, 8, and 9, as 6 + 6 + 6 + 6 + 6. Then solve the following multiplication problems for the system:

a. 6 × 8 =
b. 6 × 6 =
c. 6 × 9 =
d. 9 × 9 =
e. 5 × 9 =
f. 5 × 6 =
g. 7 × 7 =
h. 8 × 5 =



Non-Prime Modular System — New Viewpoints

Finite Mathematical Systems Simplify Matters — or Do They?

In multiplication of counting numbers in ordinary arithmetic, the answer for the problem $? \times 3 = 0$ is solved quite easily. Since the product is 0, we know that the number needed to replace the question mark must be zero. You often have heard the statement that if the product is zero one or both of the factors must be zero. This is not so! At least this is not so in certain finite arithmetic systems. How can a problem in division have *two* answers? You might think that this is never possible. In the system of clock arithmetic mod 6, however, $4 \div 4 = 1$. This might not seem strange to you, except that $4 \div 4$ also equals 4 in mod 6 arithmetic. Let's explore this finite system in more detail.

The clock arithmetic systems with a modulus of 2, 3, or 7 are prime modular systems, since 2, 3, and 7 are *prime numbers*. Numbers which have no factors except 1 or themselves are prime numbers.

Let's consider the system of clock arithmetic with non-prime modulus 6.

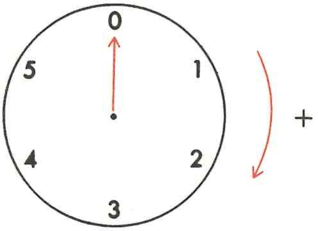


Figure 22

In mod 6, $5 + 4$ means rotate 5 spaces, then 4 spaces, ending with a sum of 3. Thus, 5 + 4 = 3. We can complete the addition table in Figure 23 with little difficulty.

		Addend					
	+	0	1	2	3	4	5
Addend	0	0	1	2	3	4	5
	1	1	2	3	4	5	0
	2	2	3	4	5	0	1
	3	3	4	5	0	1	2
	4	4	5	0	1	2	3
	5	5	0	1	2	3	4

Figure 23

What about multiplication in the mod 6 system? We need to make note of several somewhat strange patterns in the completed multiplication table in Figure 24.

		Factor					
	×	0	1	2	3	4	5
Factor	0	0	0	0	0	0	0
	1	0	1	2	3	4	5
	2	0	2	4	0	2	4
	3	0	3	0	3	0	3
	4	0	4	2	0	4	2
	5	0	5	4	3	2	1

Figure 24

When the table is symmetric with respect to the diagonal, the operation is commutative. But note the products resulting from the multiplication of 2. $2 \times 0 = 0$; $2 \times 1 = 2$; $2 \times 2 = 4$; $2 \times 3 = 0$; $2 \times 4 = 2$; and $2 \times 5 = 4$. Besides the strange result of 2×3 being equal to zero even though neither factor is zero, only answers of 0, 2, or 4 result regardless of the element multiplied by 2. Similarly, only answers of 0 or 3 result in multiplications by 3. These facts will become more significant when we consider division in this system.

No particular difficulty is encountered in the operation of subtraction. $5 - 4$ means a number n such that $n + 4 = 5$. In mod 6, $1 + 4 = 5$ and, thus, $5 - 4 = 1$. Does $5 - 4 = 4 - 5$? The answer is no, since $4 - 5 = 5$ in mod 6.

Keep Working and You Will Find the Answer — Not Always!

Have you ever worked for such a long time on a problem in mathematics that you began to wonder if an answer exists? This might seem strange to you, but some problems in mathematics cannot be solved under the given conditions.

This happens in the consideration of division in the mod 6 system of clock arithmetic. In terms of the elements of mod 6, $5 \div 1$ means a number such that when it is multiplied by 1 it gives the product 5. In general terms, $a \div b = c$ such that $c \times b = a$. In solving the problem $5 \div 1$, we see from the multiplication table that $1 \times 5 = 5$, thus, $5 \div 1 = 5$.

In the problem $4 \div 2$, a different situation confronts us. The multiplication table shows clearly that $2 \times 2 = 4$ and also $2 \times$

5 = 4. Therefore, both elements 2 and 5 satisfy the requirements for the quotient of the problem 4 ÷ 2.

But there is more! In the problem 5 ÷ 3, we are searching for a quotient n such that $n \times 3 = 5$. Look at the multiplication table for mod 6. There is no number n such that $n \times 3 = 5$. The problem 5 ÷ 3 has no answer in this system.

Check the answers in the completed division table for mod 6 in Figure 25. Relate these answers to the multiplication table developed earlier. Division by zero is excluded.

Dividend

Divisor

÷	0	1	2	3	4	5
0	—	—	—	—	—	—
1	0	1	2	3	4	5
2	0	—	1, 4	—	2, 5	—
3	0	—	—	1,3,5	—	—
4	0	—	2, 5	—	1, 4	—
5	0	5	4	3	2	1

Figure 25

EXERCISE SET 7

Operations in Non-Prime Modular Systems

1. Answer the following exercises in regard to the mod 6 system of clock arithmetic.

a. Is addition associative?

Check the following for the *associative principle.*

(1) $6 + (5 + 4) \overset{?}{=} (6 + 5) + 4$

(2) $3 + (2 + 0) \overset{?}{=} (3 + 2) + 0$

(3) $0 + (5 + 1) \overset{?}{=} (0 + 5) + 1$

(4) $5 + (5 + 5) \overset{?}{=} (5 + 5) + 5$

b. What is the identity element for addition? For multiplication?

c. Is the system distributive for multiplication with respect to addition? Subtraction?
Check these exercises for distributivity.

(1) $6(5 + 6) \overset{?}{=} 6 \cdot 5 + 6 \cdot 6$

(2) $5(3 + 0) \overset{?}{=} 5 \cdot 3 + 5 \cdot 0$

(3) $6(5 - 6) \overset{?}{=} 6 \cdot 5 - 6 \cdot 6$

(4) $5(3 - 0) \overset{?}{=} 5 \cdot 3 - 5 \cdot 0$

d. Give the additive inverses of each of the elements below.

$5 + \_\_\_\_ = 0$
$4 + \_\_\_\_ = 0$
$3 + \_\_\_\_ = 0$
$2 + \_\_\_\_ = 0$
$1 + \_\_\_\_ = 0$
$0 + \_\_\_\_ = 0$

e. Is the operation of subtraction closed in the mod 6 system?

f. Is the operation of division commutative?
Check these exercises for commutativity.

(1) $1 \div 5 \overset{?}{=} 5 \div 1$

(2) $4 \div 1 \overset{?}{=} 1 \div 4$

(3) $3 \div 5 \overset{?}{=} 5 \div 3$

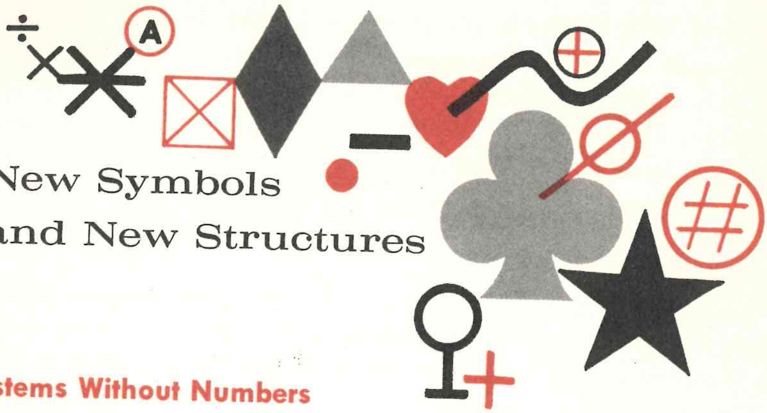
(4) $3 \div 2 \overset{?}{=} 2 \div 3$

2. Complete the table of division for the mod 4 system of clock arithmetic. List the division facts that make the mod 4 system unsuitable as an arithmetic system.

Dividend

	÷	0	1	2	3
Divisor	0				
	1				
	2				
	3				

Figure 26



New Symbols and New Structures

Sub-Systems Without Numbers

You might be surprised to learn that the elements in a mathematical system need not be numbers. Nor are the usual symbols denoting the fundamental operations $+$, $-$, $\times$, and $\div$ absolutely necessary. For example, we might use symbols such as $\oplus$, $*$, $\varnothing$, or $\sim$ to replace the symbol $+$ without changing the addition result just as long as the definition of the operation denoted by the symbol remained unchanged. Thus, if $+$ and $\oplus$ were said to represent the same operation, then $5 + 7$ and $5 \oplus 7$ would both equal 12 in the addition of counting numbers.

If E is used to represent *even numbers*, such as 2, 4, or 6; the letter O is used to represent *odd numbers*, such as 1, 3, or 5; and $\oplus$ means to add these by counting, then we can complete the addition table for this finite system as follows:

⊕	O	E
O	E	O
E	O	E

Figure 27

The addition of two odd numbers, such as $5 + 3$, results in an even number, 8. Too, the combinations of odd numbers such as $7 + 5$, $13 + 1$, $9 + 3$, and $27 + 31$ all have even numbers as sums. Therefore, we conclude $O \oplus O = E$. Does an odd or an even number result from adding two even numbers? $4 + 4 = 8$; $2 + 2 = 4$; $14 + 14 = 28$; and $16 + 2 = 18$. We conclude that

$E \oplus E = E$. What is the result of adding an odd and an even number? By such examples as $7 + 4 = 11$, $5 + 8 = 13$, and $9 + 4 = 13$, we can see that $O \oplus E = O$.

Can we apply the properties of commutativity, associativity, closure, and other properties to these systems without numbers, as in Figure 27? Is the operation $\oplus$ closed? For all elements in the system, $a + b = c$; and, therefore, the system is closed under addition.

Check the following example for the commutative principle.

Does $E \oplus O = O \oplus E$?

From the table for operation $\oplus$, $E \oplus O = O$ and $O \oplus E = O$. This example is commutative. We could replace the elements E and O with appropriate numbers.

$$\begin{aligned} E \oplus O &= O \oplus E \\ 4 \oplus 3 &= 3 \oplus 4 \\ 7 &= 7 \\ (Odd) &= (Odd) \end{aligned}$$

Is the system associative?

a.
$$\begin{aligned} O \oplus (E \oplus E) &\overset{?}{=} (O \oplus E) \oplus E \\ O \oplus E &\overset{?}{=} O \oplus E \\ O &= O \end{aligned}$$

b.
$$\begin{aligned} E \oplus (O \oplus O) &\overset{?}{=} (E \oplus O) \oplus O \\ E \oplus E &\overset{?}{=} O + O \\ E &= E \end{aligned}$$

In both instances the associative principle holds true for the system.

Is there an identity element for the operation $\oplus$?

Note: a. $E \oplus E = E$
b. $O \oplus E = O$
c. $E \oplus O = O$
d. $O \oplus O = E$

What do you notice in the results of *a*, *b*, and *c*? The combining of E with any other element always gives the element to which E is added. We can conclude that E is the additive identity.

Look at *a* and *d*. Since only $E \oplus E$ and $O \oplus O$ result in the identity element, E and O are said to be their own inverses for the operation $\oplus$.

Are you ready to explore the operation ⊕ in more detail? If so, complete a subtraction and a multiplication table for ⊕.

Abstract Systems — a Touch of Pure Mathematics

Not long ago someone stated that mathematics was the only field where nothing more remained to be done. Yet, new extensions of mathematics are increasing at a rapid rate. One extension that we might consider which pertains to finite mathematical systems is a system which is completely abstract. In a general sense, the term abstract system means a system which has no particular reference to concrete objects. Consider the following table for an operation represented by the symbol ☆ (star).

☆	M	N	O	P
M	N	O	P	M
N	O	P	M	N
O	P	M	N	O
P	M	N	O	P

Figure 28

Notice that the table in Figure 28 is symmetric with respect to the diagonal from upper left to lower right.

Does $M \star N = N \star M$?

Does $P \star O = O \star P$?

Check for associativity.

$$P \star (M \star O) \overset{?}{=} (P \star M) \star O$$

$$P \star (P) \overset{?}{=} (M) \star O$$

$$P = P$$

Since both sides are equal to P, the associative principle holds true for this example.

a. $M \star (O \star O) \overset{?}{=} (M \star O) \star O$

b. $O \star (N \star M) \overset{?}{=} (O \star N) \star M$

Notice the answers under the element *P* in Figure 28. What elements are combined with *P* to obtain these results? The identity element for the operation ☆ is ____.

Give the inverse of each of the following for operation ☆.

The inverse of

M is ____.

N is ____.

O is ____.

P is ____.

Go back to page 9 and look at the addition table for mod 7 arithmetic. By substituting the following symbols for each element of the mod 7 table and the symbol ∅ for +, the table looks like Figure 29.

0 = △

1 = ⊠

2 = ♀

3 = ♥

4 = #

5 = ×

6 = ●

∅	△	⊠	♀	♥	#	×	●
△	△	⊠	♀	♥	#	×	●
⊠	⊠	♀	♥	#	×	●	△
♀	♀	♥	#	×	●	△	⊠
♥	♥	#	×	●	△	⊠	♀
#	#	×	●	△	⊠	♀	♥
×	×	●	△	⊠	♀	♥	#
●	●	△	⊠	♀	♥	#	×

Figure 29

From Figure 29, you observe that the table is symmetric; therefore, the system is commutative for the operation ∅. The identity element is △. Does each element have an additive inverse?

Group Structure in Mathematics

The study of groups is a new field of mathematics. It is an example of how mathematics is independent of number and computation. It is an example of the logical structure of mathematics. Let's see what a mathematical group is like.

In a certain department store, three new men were hired. The employer was anxious that each man know each of the other men's jobs in case of absence or emergency and left instructions that there would be changes each week in working assignments. The three workers, Abe, Bill, and Chuck, worked on their own jobs for a week before the boss sent down the following orders: On Monday, Abe will take Bill's place, Bill will take Chuck's place, and Chuck will work in Abe's place. To re-emphasize the order, the boss made the following diagram $\begin{bmatrix} A & B & C \\ C & A & B \end{bmatrix}$. The top row of letters represented Abe (A), Bill (B), and Chuck (C). The letter under A, which was C, meant that Chuck would replace Abe. Similarly, Abe (A under B) would replace Bill, and (B under C) meant that Bill would take the place of Chuck.

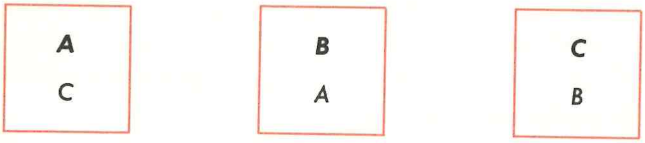
After one week with this arrangement, the boss sent another note which simply indicated the new assignment as follows: $\begin{bmatrix} C & A & B \\ B & C & A \end{bmatrix}$. The men quickly interpreted this to mean that next week Bill would replace Chuck (B under C), Chuck would replace Abe (C under A), and Abe would replace Bill (A under B). The switching of jobs was to continue for a long period of time. At lunchtime the three men would attempt to guess the arrangement for work that the boss would suggest for the coming week. It was not long before the following possibilities came to view:

$$u = \begin{bmatrix} A & B & C \\ C & A & B \end{bmatrix} \qquad v = \begin{bmatrix} A & B & C \\ B & C & A \end{bmatrix} \qquad w = \begin{bmatrix} A & B & C \\ B & A & C \end{bmatrix}$$

$$x = \begin{bmatrix} A & B & C \\ C & B & A \end{bmatrix} \qquad y = \begin{bmatrix} A & B & C \\ A & C & B \end{bmatrix} \qquad z = \begin{bmatrix} A & B & C \\ A & B & C \end{bmatrix}$$

Rather than writing out each arrangement, the letters u, v, w, x, y, and z were assigned as indicated above. Thus, if the order from the boss simply said y, the combination $\begin{bmatrix} A & B & C \\ A & C & B \end{bmatrix}$ would be used. A would remain on the same job, C would replace B, and B would replace C. The arrangement z meant no change would take place.

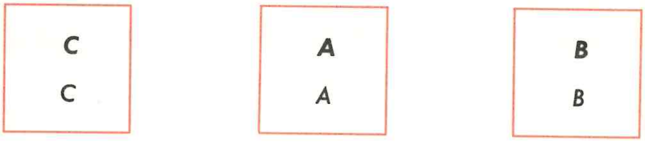
As a few months passed, the boss decided to have a little fun with the three men and started to indicate working assignments as follows: uv. This meant that the men were to find the final result or product by first performing arrangement u, then arrangement v. Similarly, uw meant to work on the jobs resulting from doing arrangement *u*, $\begin{bmatrix} A & B & C \\ C & A & B \end{bmatrix}$, followed by performing arrangement *w*, $\begin{bmatrix} A & B & C \\ B & A & C \end{bmatrix}$. Thus, to figure the week's working arrangement for *uw*, the men reasoned as follows: *uw* means $\begin{bmatrix} A & B & C \\ C & A & B \end{bmatrix} \begin{bmatrix} A & B & C \\ B & A & C \end{bmatrix}$ (perform *u*, then perform *w*). In arrangement *u*, *C* replaces *A*, *A* replaces *B*, and *B* replaces *C*. The result for the first part would be charted as follows:



C replaces A　　A replaces B　　B replaces C

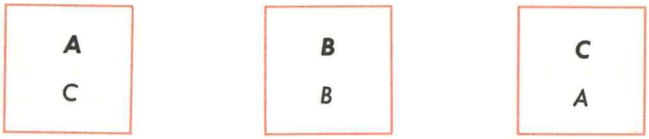
Now *w* must be performed. In arrangement *w*, *B* replaces *A*, *A* replaces *B*, and *C* remains unchanged. Thus, *uw* would mean performing *u*, then performing *w*:

I (*u*)



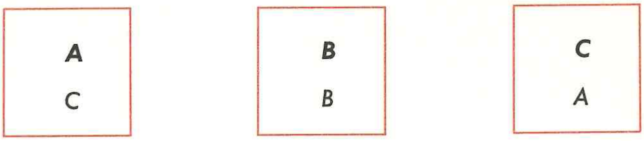
I. (C replaces A, A replaces B, and B replaces C.)

II (*w*)



II. (Performing *w* means: C remains unchanged, B replaces A, and A replaces B.)

End Result



(C has replaced A, B was unchanged, and A replaced C.)

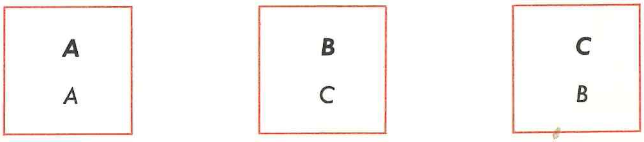
In the *end result* of performing uw, C has replaced A, B has remained unchanged, and A has replaced C; this final arrangement is $\begin{bmatrix} A & B & C \\ C & B & A \end{bmatrix}$, or arrangement x. In other words, $uw = x$.

Review this operation once again and then find the result of yx. Study the solution to yx which follows.

Step 1 $\quad yx = \begin{bmatrix} A & B & C \\ A & C & B \end{bmatrix} \begin{bmatrix} A & B & C \\ C & B & A \end{bmatrix}$

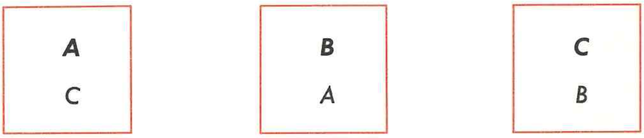
Step 2 $\quad A$ remained the same, C replaced B, and B replaced C.

Result of Step 2



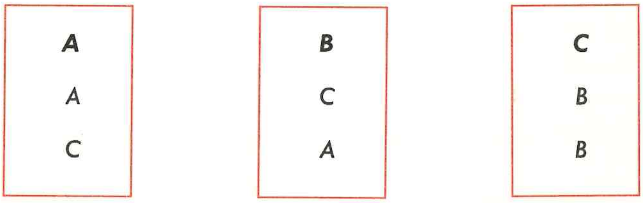
Step 3 $\quad$ In the result of y, replace A by C, let B remain the same, and replace C by A, as indicated by arrangement x.

Result of Step 3



In the final result of yx, C has replaced A, A has replaced B, and B has replaced C. (See the following diagram.)

yx



Therefore,

$$yx = \begin{bmatrix} A & B & C \\ C & A & B \end{bmatrix}, \text{ or } u, \qquad \text{and} \qquad yx = u.$$

Work the following problems for the operation.

a. $uu =$
$vu =$
$wu =$
$xu =$
$yu =$
$zu =$

b. $uv =$
$vv =$
$wv =$
$xv =$
$yv =$
$zv =$

Complete the following operation table.

Second Arrangement

First Arrangement

	u	v	w	x	y	z
u	v	z	x		w	u
v	z	u	y		x	v
w	y		z	v	u	w
x	w		u	z	v	x
y	x	w	v		z	y
z	u					z

Figure 30

Note the following facts about basic laws that have been discussed in this booklet in regard to the operation just discussed.

a. Does the commutative law hold for the operation, that is, does $yw = wy$? Check other possibilities such as $xy = yx$ and $uv = vu$ for commutativity. Although the commutative law fails for this operation, note the cases in which it does hold true. In other words

$$vu = uv$$

b. Does the associative law hold, that is, does $u(vw) = (uv)w$? (Perform vw, then perform this result after doing u. Note that

$vw = y$ and $uy = w$. Now check to see if the right-hand member equals w.)

Solution check.

$$u(vw) \overset{?}{=} (uy)w$$
$$uy \overset{?}{=} zw$$
$$w = w$$

Since both sides result in the answer w, associativity holds for this example. Check the following for associativity.

$$w(xy) \overset{?}{=} (wx)y$$
$$y(wz) \overset{?}{=} (yw)z$$
$$u(wu) \overset{?}{=} (uw)u$$

c. For every two arrangements is there one and only one result which is an element in the set? Since this is true, the set is closed for the operation.

d. Is there an identity or neutral element? Note $yz = y$, $wz = w$, $uz = u$, $xz = x$, etc. The identity element is ___?___.

e. Is there an inverse element for every element in the set? Note also that $yy = z$, $vu = z$, $xx = z$, $uv = z$; that is, for each element there is an inverse element such that when the two elements are combined by the operation the result is the identity element z.

Hence, the operation follows the laws of closure and associativity. It has an identity element. Each element has an additive inverse. Yet this operation is not commutative.

When *two* elements of a set are combined under an operation so that another element in the set results, we say it is a *binary operation*.

Now we can define the properties of a group. A *mathematical group* is defined as a set of elements with a given binary operation which obeys the following properties:

1. The set is closed under the operation.
2. The elements obey these conditions:
   a. There is an identity element.
   b. The operation is associative.
   c. For each element there is an inverse element.

If the group obeys the commutative property, we say it is an *Abelian group*.

EXERCISE SET 8

Considering a Group

1. Using the information from the preceding section, find the result of the following:
   - a. $wx =$
   - b. $zx =$
   - c. $yu =$
   - d. $uy =$
2. Show that the operation wx is not commutative.
3. Show that $w(xy)$ is associative; that is, show that $w(xy) = (wx)y$.
4. What is the identity element for the operation? Explain.

Mathematical Symbols — What Do They Stand For?

Other interesting group structures can be developed logically through movement of objects. Consider a given rectangle A in a regular position as shown in Figure 31.

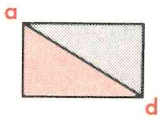
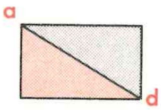
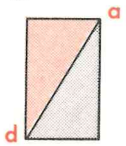


Figure 31

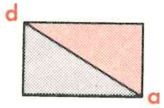
Other positions in which we can place the rectangle by clockwise rotations of quarter-turns are shown in Figure 32.



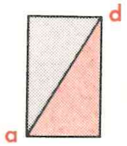
performing M on A
(no turn)



performing S on A
(three quarter-turns)



performing N on A
(two quarter-turns)

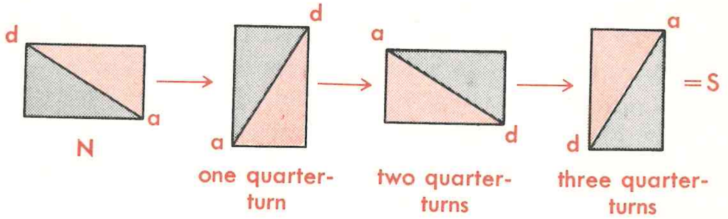


performing B on A
(three quarter-turns)

Figure 32

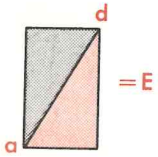
Our group consists of four elements *M*, *S*, *N*, and *E* under the binary operation #. *M* # *S* is defined as the result of performing *M*, then *S*.

Follow the steps in the solution of the problem *N* # *E*. We find the solution by performing *N* on the original rectangle *A*, then performing *E*. *E* represents three quarter-turns.

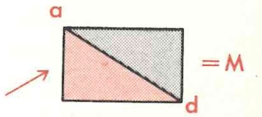


The final position of the rectangle represents the position *S*. The answer, then, is *N* # *E* = *S*.

Consider another example. What does *E* # *S* equal?

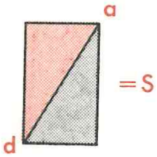


S represents one quarter-turn.
Rotate *E* one quarter-turn.



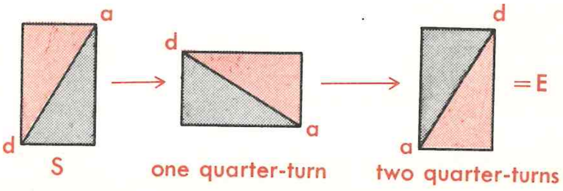
Therefore, *E* # *S* = *M*.

Try to solve the problem *S* # *N* on your own; then check your work with the solution illustrated below.



N represents two quarter-turns.

Therefore, rotate *S* two quarter-turns.



Thus, by performing *S*, then *N*, we get the result *E*.

EXERCISE SET 9

Extending the Group Concept

1. Study the completed table below for the operation #⃝ on rectangles.

Second Operation

First Operation

#⃝	M	S	N	E
M	M	S	N	E
S	S	N	E	M
N	N	E	M	S
E	E	M	S	N

Figure 33

a. Is the operation #⃝ closed?

b. Is the operation #⃝ commutative? Does *E* #⃝ *S* = *S* #⃝ *E*? Is there symmetry in the table?

2. Is the operation #⃝ associative? Does *M* #⃝ (*E* #⃝ *S*) = (*M* #⃝ *E*) #⃝ *S*?

3. Is there an identity element?

4. Does each element have an inverse? List the inverse of each element in the group.

The inverse of

M is ______. *N* is ______.
S is ______. *E* is ______.

5. Consider the equilateral triangle in Figure 34.

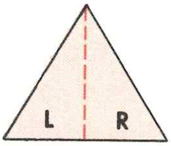


Figure 34

Let N represent no change in the position of the card. A change in position of the triangle made by rotating the triangle about the dotted line 180 degrees will be represented by D.

$N \sim D$ means to perform N then D. The net result leaves the triangle in position D and $N \sim D = D$. Complete the table in Figure 35 for the operation with elements N and D.

Second Operation

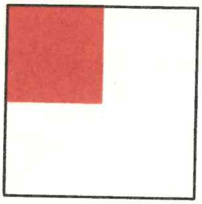
First Operation

~	N	D
N		D
D		

Figure 35

6. Does the group in exercise 5 represent an Abelian group?

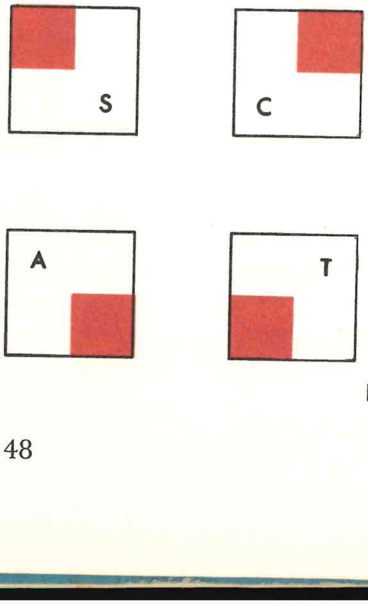
7. Position A represents the original position of a square with a marked corner as shown below.



A

Figure 36

S will represent no change in position A, *C* will represent a rotation of 90 degrees or one quarter-turn of the square clockwise, *A* represents two rotations, and *T* represents 3 rotations. *C* # *A* means to perform *C* on the original square *A*, then perform *A*. Complete the table for the operation #

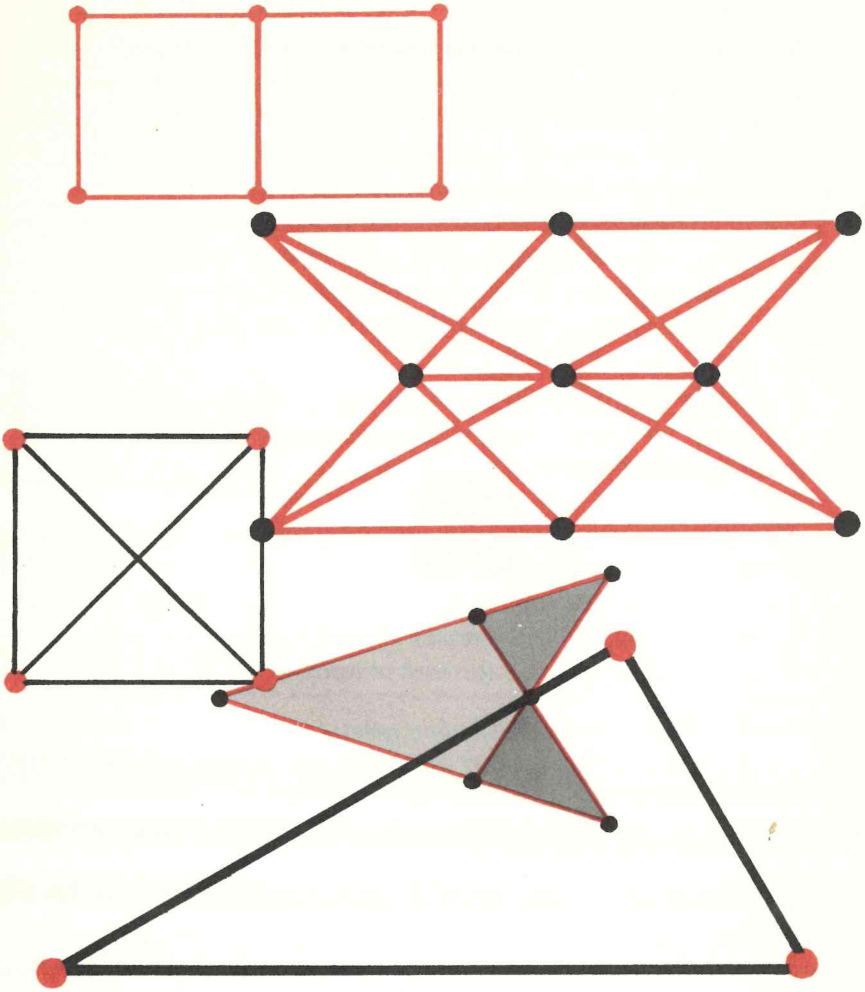


Second Operation

First Operation

#	S	C	A	T
S	S			
C		A		
A			S	
T				A

Figure 37



Finite Geometry — a New Consideration

The Skeleton of a Geometry

In the same sense that a finite number system has a specific number of elements, we can develop a finite geometry which has a definite number of elements. These elements can be points, lines, or planes.

You might have heard the statement that mathematics is a logical science. Just what does this statement really mean? A logical science begins with undefined terms and certain assumptions that are accepted without proof. From these terms and assumptions, new terms and operations are defined. Then a sequence of statements based on these assumptions and definitions is used to prove whether or not a statement is valid. Thus, the definitions and assumptions we make at the outset determine the nature of the system. We use this idea in the development of our finite geometry.

A Three-Point Geometry

Can you imagine a geometry which has only 3 points? Let's see what this finite geometry is like.

In developing this logical finite system, we will use the undefined terms of *point*, *line*, *point on a line*, and *line on a point.*

Next, let's assume the following axioms.

A_1 There exist exactly three points.
A_2 On any two distinct points there is exactly one line.
A_3 Not all points are on the same line.
A_4 On any two distinct lines there is at least one common point.

We can draw a figure which might be helpful in our work. We must keep in mind, however, that the proofs we make in the system depend solely upon our definitions, and the four assumed axioms along with other theorems which are logically developed.

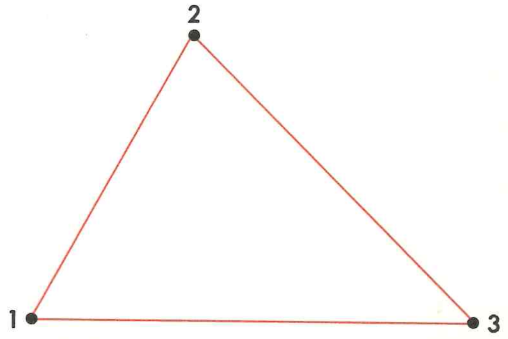


Figure 38

Let's consider the following theorem.

Theorem 1 — On any two distinct lines there is not more than one point in common.

Let us use the notation (1,2) to designate a line. That is, a line may be identified by listing the points on that line. By inspection of Figure 38, we see that point 1 is the only point common to lines (1,3) and (1,2). However, we cannot rely on our examination of the figure as a proof even though this is helpful.

By A_1, there are exactly three points. Let us call them points 1, 2, and 3. Since by A_3, at least one point is not on a given line, only two points may be on a given line. Let (1,2) be a given line. There is a third point, 3, that is not on (1,2). Any other line must include 3 as one of its two points. Since only two points may be on a line, the other line must include 1 or 2, but not both. Thus, if (1,2) is a line, any other line can include only one of the two points. So we can say that Theorem 1 has been proven true.

Consider Theorem 2 in our three-point geometry.

Theorem 2 — There exist three and only three distinct lines.

Axioms 1, 2, and 3 state respectively that there are exactly three points; on any two distinct points there is exactly one line; and not all points are on the same line. What conclusion can we reach for Theorem 2? If there are exactly three points by Theorem 1, point 1 is common to only two lines, say (1, 3) and (1, 2); point 2 is common to lines (1, 2) and (2, 3); and point 3 is common to lines (1, 3) and (2, 3). Thus, the three points determine but three different lines, namely (1, 2), (1, 3), and (2, 3).

Theorem 3 — Not all lines are on the same point.

By A_3, not all points are on the same line. Thus, given any line, there is a point not on it. This point is on at least one line. Therefore, at least one line (the given line) is not on the point. ("Not all lines are on the same point" is equivalent to saying "at least one line is not on a given point.")

Duality — When Changing Does Not Change!

A remarkable property in the three-point finite geometry system is that of *duality*. It is a property possessed by this finite geometry which is both interesting and timesaving.

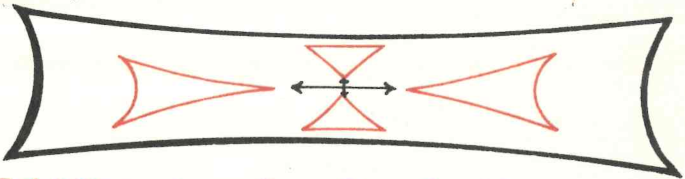
Consider Axiom 1 of the three-point geometry. **A_1 There exist exactly three distinct *points*.**

In Theorem 2, we proved that in the system there also existed exactly three distinct *lines*. If in Axiom 1 we substituted the word *lines* for the word *points*, the statement remains true. It is said that Axiom 1 and Theorem 2 are *dual* statements.

Is there a dual statement for Axiom 2? By changing the word *points* to *lines* and the word *line* to *point*, Axiom 2 reads as follows: **On any two distinct lines there is exactly one point.** Is this a valid statement for our three-point geometry? Actually, we proved this earlier as Theorem 1 in the system.

In dual relationships, we can interchange the elements *point* and *line*, and *point* and *plane*. For example, the dual of the statement which says that not all points are on the same line will say that not all lines are on the same point.

In regard to our finite three-point geometry, Theorems 2, 1, and 3 are duals of Axioms 1, 2, and 3 respectively. The dual of any statement which has been proved valid does not require that we prove it, too.



A Four-Point Geometry — Some New Considerations

We are now going to construct a new miniature geometry which has four points. Although mathematicians are not interested primarily in concrete applications of the systems that they might invent, we will find that our finite geometry has concrete interpretations which can be applied.

This four-point geometry will have the undefined terms of *point*, *line*, *plane*, *point on a line*, *line on a point*, and *point on a plane*. The term **parallel** will be defined as follows: **If L_1 and L_2 are lines, then L_1 is parallel to L_2 if, and only if, L_1 and L_2 have no point in common.**

Our geometry has five axioms stated as follows:

A_1 There exist exactly four distinct points.

A_2 On any two distinct points there is exactly one line.

A_3 On any line there are exactly two points.

A_4 On any three points there is exactly one plane.

A_5 On any plane there are exactly three points.

Since there are exactly four points, you might conclude there are only four lines. Let's prove Theorem 1 for this finite geometry which states there are *six* lines.

Theorem 1 — There exist exactly six distinct lines.

Label the four points 1, 2, 3, and 4. Axiom 2 says that on any two distinct points there is exactly one line. The possibilities for lines, therefore, are represented by the different point combinations: (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), and (3, 4). It is apparent that Theorem 1 is valid.

Figure 39 might be helpful in checking this result.

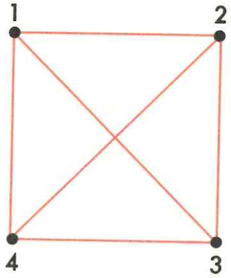


Figure 39

Consider Theorem 2 in our four-point geometry.

Theorem 2 — On any point not on a given line there is exactly one line parallel to the given line.

We are given one point, say point 1, and a line (2, 4) not on point 1. We must prove that there is exactly one distinct line on point 1 parallel to line (2, 4).

From the definition given for the term parallel, L_1 and L_2 are parallel if they have no point in common. By Axiom 3, on any given line there are exactly two points. We know that point 1 is not on line (2, 4). By Axiom 3, point three cannot be on line (2, 4). Therefore, line (1, 3) is parallel to line (2, 4) by our given definition of parallel lines. To complete the proof, we simply would show that the remaining four lines in our system have a point in common with given line (2, 4).

Theorem 3 — There exist exactly four planes.

You will recall that Axioms 4 and 5 introduced a new term, *plane*. Plane is an undefined term in our miniature geometry. From axioms 4 and 5, it is quite clear that three points determine a plane in this system.

Of the four points 1, 2, 3, and 4 in our system, the following combinations of three points are possible: (1, 2, 3), (1, 2, 4), (2, 3, 4), and (1, 3, 4). Check this result by examination of Figure 39.

With only four possible combinations of three points in the system, we have proof that Theorem 3 is valid.

Now let's consider a concrete interpretation of the four-point geometry. Suppose we wanted to choose teams for games of ping-pong. By replacing the two words *points* and *lines* by the words *boys* and *teams* respectively in Theorem 1 and Axioms 1, 2, and 3, an interesting interpretation of our system is revealed.

Axiom 1 — There exist exactly four distinct *boys*. Let's name these boys Abe, Bill, Chuck, and Dave.

Theorem 1 — There exist exactly six *teams*.

Axiom 3 — On any *team* there are exactly two *boys*.

Team 1	(Abe, Bill)
Team 2	(Abe, Chuck)
Team 3	(Abe, Dave)
Team 4	(Bill, Chuck)
Team 5	(Bill, Dave)
Team 6	(Chuck, Dave)

By a slight change in the wording of Axiom 2 and Theorem 2, we can verify them in this interpretation.

Axiom 2 — On (for) any two distinct *boys* there is exactly one *team*.

Theorem 2 — On (for) any given *boy* not on a given *team* there is exactly one *team* parallel to the given team.

We must interpret Theorem 2 in light of the definition of parallel. In other words, we are looking for a team which has no boy in common with the given team.

Let's select Dave as the *boy* in Theorem 2 and team 1 (Abe and Bill) as the given *team*.

Of the other five teams, is there a team of which Dave is a member but either Abe or Bill is not? Dave is a member of teams 3, 5, and 6. But we find that either Abe or Bill is also common to teams 3 and 5. Team 6 (Dave and Chuck) represents the verification of Theorem 2.

There is really no end to the interesting challenges which finite geometries can provide. In this booklet we can explore only a few of the wonders of this topic. You might want to consider what a six-point or seven-point geometry is like.

? Points and ? Lines

We have considered a three-point and a four-point finite geometry. Now let's consider a finite geometry where the number of points or lines are not given to us immediately. How can we determine the specific number of points and lines in the geometry? This will depend upon the assumed axioms and given terms. Suppose we are given the undefined terms of *point*, *line*, *point on a line*, and *line on a point* and the defined term *parallel*. L_1 is parallel to L_2 if, and only if, there is no point which is on both L_1 and L_2.

The following four axioms are assumed.

A_1 There is at least one point.

A_2 Every line consists of exactly two distinct points which are on that line.

A_3 Every point is on exactly two distinct lines.

A_4 For any given line there are exactly three other distinct lines which are parallel to the given line.

Let's begin the exploration of this finite system by considering Theorem 1.

Theorem 1 — There is at least one line.

By Axiom 1, there is at least one point. By Axiom 3 we know that every point is on exactly two lines and therefore Theorem 1 follows logically.

Theorem 2 — To a given line there are exactly two other non-parallel lines.

Recall that L_1 and L_2 are parallel if, and only if, there is no point which is on both L_1 and L_2. Which axiom in our geometry might prove most helpful? Axiom 4 tells us there are exactly three lines parallel to any given line.

Suppose the given line in Theorem 2 is on distinct points (1, 2). What might we conclude concerning Axiom 4 and line (1, 2)? There must be three distinct lines which have no points in common with line (1, 2), say lines with points (3, 4), (4, 5), and (5, 6). Axiom 3 states clearly that every point is on two distinct lines.

We have established that there are at least four lines.

$L_1 = (1, 2)$
$L_2 = (3, 4)$
$L_3 = (4, 5)$
$L_4 = (5, 6)$

We need two more lines with points (2, 3) and (1, 6) in order to keep from violating Axiom 3.

$L_5 = (2, 3)$
$L_6 = (1, 6)$

With the completion of L_5 and L_6, we have verification of Axiom 3; each point is on two distinct lines.

Two other theorems which we now can prove with little difficulty will answer the questions — how many points and how many lines?

Theorem 3 — There are exactly six lines.

Axiom 4 and Theorem 2 give us needed proof for Theorem 3. A given line has exactly 3 parallel lines and exactly 2 non-parallel lines. Thus, we have a total of 6 lines.

Theorem 4 — There are exactly six points.

Actually, we established Theorem 4 as we considered a proof for Theorem 2. From a given line on points (1, 2), we have verified the existence of parallel lines with points (3, 4), (4, 5), and (5, 6). Non-parallel lines had points (2, 3) and (1, 6). In all, we have a six-point geometry with points 1, 2, 3, 4, 5, and 6.

Study Figure 40 for this six-point geometry and verify each of the four axioms and four theorems we have considered.

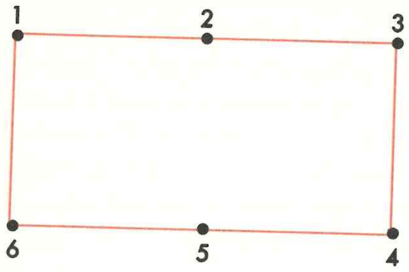


Figure 40

EXERCISE SET 10

Extensions of Finite Geometries

1. State in your own words what is meant by the statement that mathematics is a logical science.

2. Using the terms *point*, *line*, *point on a line*, and *line on a point* as undefined terms and axioms 1 through 6 listed below, prove theorems 1 and 2 for this nine-point geometry.

A_1 On any two distinct points there is at most one line.
A_2 Not all points are on the same line.
A_3 There exists at least one line.
A_4 On any line there are at least three distinct points.
A_5 On any line there are at most three distinct points.
A_6 On any given point not on a given line there is exactly one line parallel to the given line.

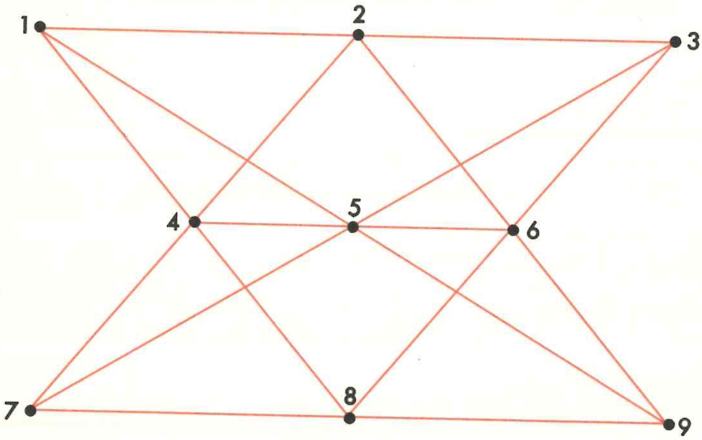


Figure 41

Theorem 1 — On any two distinct lines there is at most one common point.

Theorem 2 — There exist at least 3 points.

3. Team 1 (Abe, Dave)
Team 2 (Abe, Chuck)
Team 3 (Chuck, Dave)

Using the concrete interpretation above of a finite geometry, check to see if the following three axioms are true by inserting the words *boys* and *teams* for the words *points* and *line* respectively in each axiom.

Axiom 1 — **Every two lines have one and only one point in common. (Every two *teams* have one and only one *boy* in common.)**

Axiom 2 — **Every point lies on two and only two lines. (Every *boy* lies (is) on two and only two *teams*.)**

Axiom 3 — **There are three and only three points.**

4. Given axioms 1, 2, and 3 below, prove theorems 1 and 2 with respect to this finite geometry.

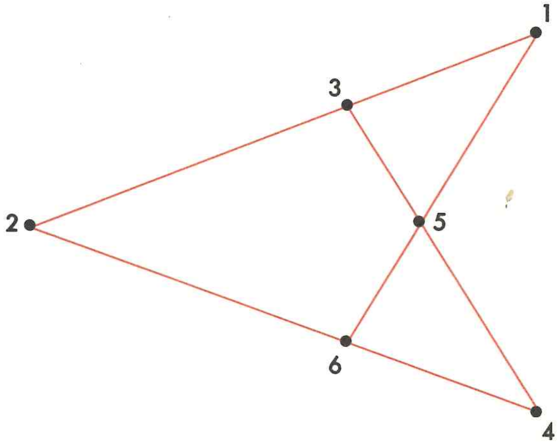


Figure 42

Definition: **Two lines are parallel if they have no point in common.**

Axiom 1 — **Every two distinct lines have one and only one point in common.**

Axiom 2 — **Every distinct point lies on only two lines.**

Axiom 3 — **There are exactly four distinct lines.**

Prove:

Theorem 1 — There are only six distinct points.

Theorem 2 — There are no parallel lines.

A Backward Glance and a Look Ahead

This booklet has given you a notion of the logical structure of mathematics. It has been illustrated how a mathematical structure can be abstract and without reference to real situations. It has been illustrated how we can create new operations to find new systems and groups.

This booklet certainly does not exhaust all of the interesting and important ideas on finite mathematical systems. We have seen that not just one number system, but many number systems, exist. The nature of any one mathematical system depends upon the definitions and assumptions made and accepted from the beginning.

Many of the *absolute laws* of mathematics with which we are so familiar are not true in other number systems. For example, we learned that one or both factors need *not* be zero to obtain a zero product in mod 6 arithmetic. For in this system, $2 \cdot 3 = 0$ and neither factor is zero. Also, in the mod 6 system, an answer for $1 \div 2$ does not exist while $2 \div 2$ has two answers: 1 and 4. Too, operation symbols such as $+$, $-$, $\times$, $\div$, $\oplus$ take on the meanings we give them by definition.

Although our study of groups may seem impractical, scientists are finding them very helpful in finding new facts. For example, along with the fact that the rings in the atom can be related to modular number systems, groups have practical research applications in such fields as agriculture.

Now that you have completed this book satisfactorily, you are urged to find more material involving number systems.

Extending Your Knowledge

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Freund, John E., *A Modern Introduction to Mathematics.* Prentice-Hall, Inc., 1956.

Kasner, E., and Newman, J., *Mathematics and the Imagination.* Simon and Schuster, 1940.

Kemeny, John G., Snell, J. Laurie, and Thompson, Gerald L., *Introduction to Finite Mathematics.* Prentice-Hall, Inc., 1957.

Richardson, M., *Fundamentals of Mathematics.* The Macmillan Company, 1958.

Solutions to the Exercises

EXERCISE SET 1

1.

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4	5	6	7	8	9	10	11	12	1	2
3	4	5	6	7	8	9	10	11	12	1	2	3
4	5	6	7	8	9	10	11	12	1	2	3	4
5	6	7	8	9	10	11	12	1	2	3	4	5
6	7	8	9	10	11	12	1	2	3	4	5	6
7	8	9	10	11	12	1	2	3	4	5	6	7
8	9	10	11	12	1	2	3	4	5	6	7	8
9	10	11	12	1	2	3	4	5	6	7	8	9
10	11	12	1	2	3	4	5	6	7	8	9	10
11	12	1	2	3	4	5	6	7	8	9	10	11
12	1	2	3	4	5	6	7	8	9	10	11	12

2. *a.* 144
   b. Yes

3. *a.* Yes, $1 = 1$
   b. Yes, $8 = 8$
   c. Yes, $10 = 10$
   d. Yes, $4 = 4$
   e. Yes, $2 = 2$

EXERCISE SET 2

1. *a.* Yes, $12 = 12$
   b. Yes, $8 = 8$
   c. Yes, $2 = 2$
   d. Yes, $8 = 8$

2. *a.* 12
   b. $12 + 12 = 12$
   $5 + 12 = 5$
   $11 + 12 = 11$
   $9 + 12 = 9$

3. *a.* 11
   b. 10
   c. 9
   d. 7
   e. 2; Yes

4. *a.*

+	0	1
0	0	1
1	1	0

b. 0

5. *a.* Yes, *b.* commutative, *c.* (1) $1 = 1$, (2) $1 = 1$, (3) $0 = 0$, (4) $1 = 1$, (5) $1 = 1$, *d.* 0, *e.* Yes, (1) 0, (2) 1, (3) 0, (4) 1

6.

Ⓢ	1	2	3	4
1	2	3	4	1
2	3	4	1	2
3	4	1	2	3
4	1	2	3	4

EXERCISE SET 3

1. *a.* Yes, *b.* It is commutative. 2. *a.* $2 = 2$, *b.* $3 = 3$, *c.* $1 = 1$, *d.* $2 = 2$, *e.* $5 = 5$

3. 1 4. *a.* 6, *b.* 3, *c.* 2, *d.* 5, *e.* 4, *f.* 1

5. *a.* $4 = 4$, *b.* $2 = 2$, *c.* $0 = 0$, *d.* $2 = 2$, *e.* $3 = 3$, *f.* $2 = 2$

6.

Addend

Addend	+	1	2	3	4	5
	1	2	3	4	5	1
	2	3	4	5	1	2
	3	4	5	1	2	3
	4	5	1	2	3	4
	5	1	2	3	4	5

Factor

Factor	×	1	2	3	4	5
	1	1	2	3	4	5
	2	2	4	1	3	5
	3	3	1	4	2	5
	4	4	3	2	1	5
	5	5	5	5	5	5

7. *a.* Modulo 5, *b.* Yes, identity element is 5, *c.* Yes, identity element is 1.

EXERCISE SET 4

1. *a.* 1
   b. 6
   c. 6
   d. 1
   e. 3
   f. 4
   g. 0
   h. 4

Subtrahend

	—	0	1	2	3	4	5	6
Minuend	0	0	6	5	4	3	2	1
	1	1	0	6	5	4	3	2
	2	2	1	0	6	5	4	3
	3	3	2	1	0	6	5	4
	4	4	3	2	1	0	6	5
	5	5	4	3	2	1	0	6
	6	6	5	4	3	2	1	0

2. Yes
3. *a.* $4 \neq 2$, *b.* $3 \neq 0$
4. *a.* Yes, $5 = 5$, *b.* Yes, $3 = 3$, *c.* Yes, $5 = 5$, *d.* Yes, $4 = 4$
   e. The system is distributive for multiplication with respect to subtraction.

5.

Subtrahend

	—	0	1	2	3	4	5
Minuend	0	0	5	4	3	2	1
	1	1	0	5	4	3	2
	2	2	1	0	5	4	3
	3	3	2	1	0	5	4
	4	4	3	2	1	0	5
	5	5	4	3	2	1	0

EXERCISE SET 5

1.

Dividend

	÷	0	1	2	3	4	5	6
Divisor	0	–	–	–	–	–	–	–
	1	0	1	2	3	4	5	6
	2	0	4	1	5	2	6	3
	3	0	5	3	1	6	4	2
	4	0	2	4	6	1	3	5
	5	0	3	6	2	5	1	4
	6	0	6	5	4	3	2	1

2. *a.* Yes, if division by 0 is excluded.
   b. No, $6 \div 4 = 5$; $4 \div 6 = 3$
   c. (1) No, $5 \neq 6$; Yes, $2 = 2$
   The operation of division is not associative as shown by example 1.
3. *a.* No, $1 \neq 5$; *b.* No, $1 \neq 6$; *c.* Yes, $0 = 0$
   Not distributive for multiplication with respect to division as shown by a and b.

EXERCISE SET 6

1.

Addend

	+	5	6	7	8	9
Addend	5	5	6	7	8	9
	6	6	7	8	9	5
	7	7	8	9	5	6
	8	8	9	5	6	7
	9	9	5	6	7	8

2. Yes
3. *a.* Yes, *b.* Yes
4. 5
5. Yes, inverse of: 5 is 5, 6 is 9, 7 is 8, 8 is 7, 9 is 6.
6. *a.* 8, *b.* 6, *c.* 5, *d.* 5,
   e. 9, *f.* 6, *g.* 9, *h.* 5

EXERCISE SET 7

1. *a.* Yes, (1) 1 = 1, (2) 5 = 5, (3) 0 = 0, (4) 3 = 3
   b. (1) 0, (2) 1
   c. Yes, (1) 0 = 0, (2) 3 = 3, (3) 4 = 4, (4) 3 = 3
   d. (1) 1, (2) 2, (3) 3, (4) 4, (5) 5, (6) 0
   e. Yes
   f. No, (1) 5 = 5, (2) 4 = –, (3) 3 = –, (4) – = –. These combinations have no answer.

2.

Dividend

	÷	0	1	2	3
Divisor	0	–	–	–	–
	1	0	1	2	3
	2	0,2	–	1,3	–
	3	0	3	2	1

EXERCISE SET 8

1. *a.* $wx = v$, *b.* $zx = x$, *c.* $yu = x$, *d.* $uy = w$
2. Any examples such as 1c and 1d where $yu \neq uy$ show that the operation is not commutative.
3. $w(xy) = (wx)y$
   $wv = vy$
   $x = x$
4. $vz = v$, $wz = w$, $zw = w$, $zx = x$

EXERCISE SET 9

1. *a.* Yes, *b.* Yes, $M = M$; *c.* Yes
2. *a.* Yes, *b.* Yes, $M = M$
3. Yes, M
4. *a.* Yes, *b.* M, E, N, S
5.

~	N	D
N	N	D
D	D	N

6. Yes
7.

#	S	C	A	T
S	S	C	A	T
C	C	A	T	S
A	A	T	S	C
T	T	S	C	A

EXERCISE SET 10

1. Mathematics, like other logical sciences, begins with certain undefined terms and assumptions. From these terms and assumptions, new terms and operations are defined. A sequence of statements based on these assumptions and definitions are used to prove whether or not a statement is valid. The nature of the system is determined by the definitions and assumptions made at the outset.

2. Theorem 1 — On any two distinct lines there is at most one common point.
Proof: By Axiom 1, on any two distinct points there is at least one line. This axiom would be violated if two distinct lines had *more* than one common point. Thus Theorem 1 is proven true.

Theorem 2 — There exist at least three points.
Proof: By Axiom 3, we know there is at least one line. Axiom 4 tells us that on any line there are at least three distinct points. Thus Theorem 2 is known to be true.

3. Yes. They are true.

4. *a*. Theorem 1 — There are only six distinct points.
Proof: We know by Axiom 3 that there are only four distinct lines. Call these lines A, B, C, and D. By Axiom 1, every two distinct lines have one and only one point in common. Four different lines can be combined in the following ways: AB, AC, AD, BC, BD, CD. In view of Axiom 1, these combinations also can represent six points. If we had more than six points, it would contradict Axiom 2. These six points are also all different. Our Axiom 2 would not hold, for example, if any of the six points were not distinct. If all six points were not distinct, there could be a point which was on more lines than permissible by Axiom 2.

b. Theorem 2 — There are no parallel lines.
Proof: By Axiom 1, we know that every pair of lines has one (and only one) point in common. Therefore no lines are parallel since this violates the definition of parallel.

Exploring Mathematics on Your Own

Exploring Mathematics on Your Own is a fascinating series of enrichment booklets tailored for students who want to go beyond the textbook. Each booklet in the series presents a stimulating topic in an informal, easy-to-read style. Each booklet is complete in itself and may be used by students for individual work or by the class as a whole. Ample practice is provided.

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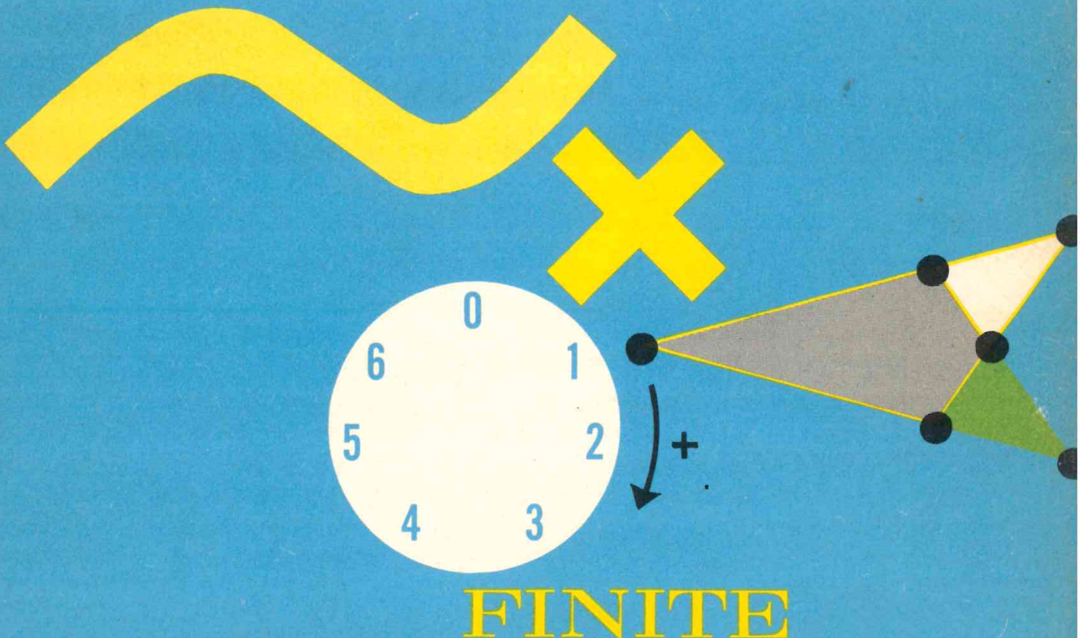

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